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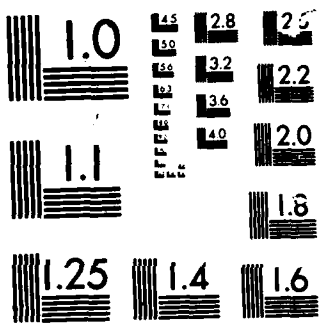

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U.S. ARMY INTELLIGENCE CENTER AND SCHOOL
Software Analysis and Management System

AD-A166 581

Testing and Combination of Confidence Ellipses: A Geometric Analysis

EAAF

Technical Memorandum No. 2

August 5, 1985

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This is one of a series of algorithm analysis reports performed for the US Army Intelligence Center and School covering selected algorithms in existing or planned Intelligence and Electronic Warfare (IEW) systems. This report proves several properties of the ellipse combination process as used in correlation algorithms, both alone and in combination with the associated statistical tests. In particular, it is shown that the statistical tests introduce order dependence to the combination process.

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PREFACE

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NO. 40

TESTING AND COMBINATION OF CONFIDENCE ELLIPSES
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5 AUGUST 1985

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I. INTRODUCTION

A confidence ellipse, sometimes called an EEP, is one of the ways in which intelligence data concerning an emitter's location is summarized. Such an ellipse is always centered at the estimated emitter location ~~as shown in Figure 1,~~ and is a measure of the possible error in the location estimate.

In Figure 1, X is the estimated location of the emitter. The ellipse is a 95% confidence ellipse, which means that an observer may be 95% confident that the ellipse contains the true location of the emitter. Put another way, it is unlikely that the estimated location is in fact the true location. The confidence ellipse defines a region in which the true location is most likely to be found. Consequently, the size of the ellipse is a measure of how good the location estimate is.

There are two ways of specifying the shape of a confidence ellipse: geometrically and statistically. The first method is that used in Figure 1. The ellipse has a semi-major axis of length A and a semi-minor axis of length B, and is rotated θ degrees from true north. In this example the axis lengths are measured in kilometers. The second method is not as easy to interpret geometrically, but is mathematically more convenient. Here the shape and size of the ellipse is defined by a <u>variance-covariance matrix</u>, which is commonly denoted by the Greek letter Σ. Since this symbol is also commonly used as the mathematical summation operator, the variance-covariance matrix will be denoted by the letter S. S is generally assumed to be known for any given sensor system.

Once an analysis center has received data in this form, a step called self-correlation is commonly performed. The goal of self-correlation is to determine if two or more pieces of data actually concern one individual emitter. It is clear why this is desirable.

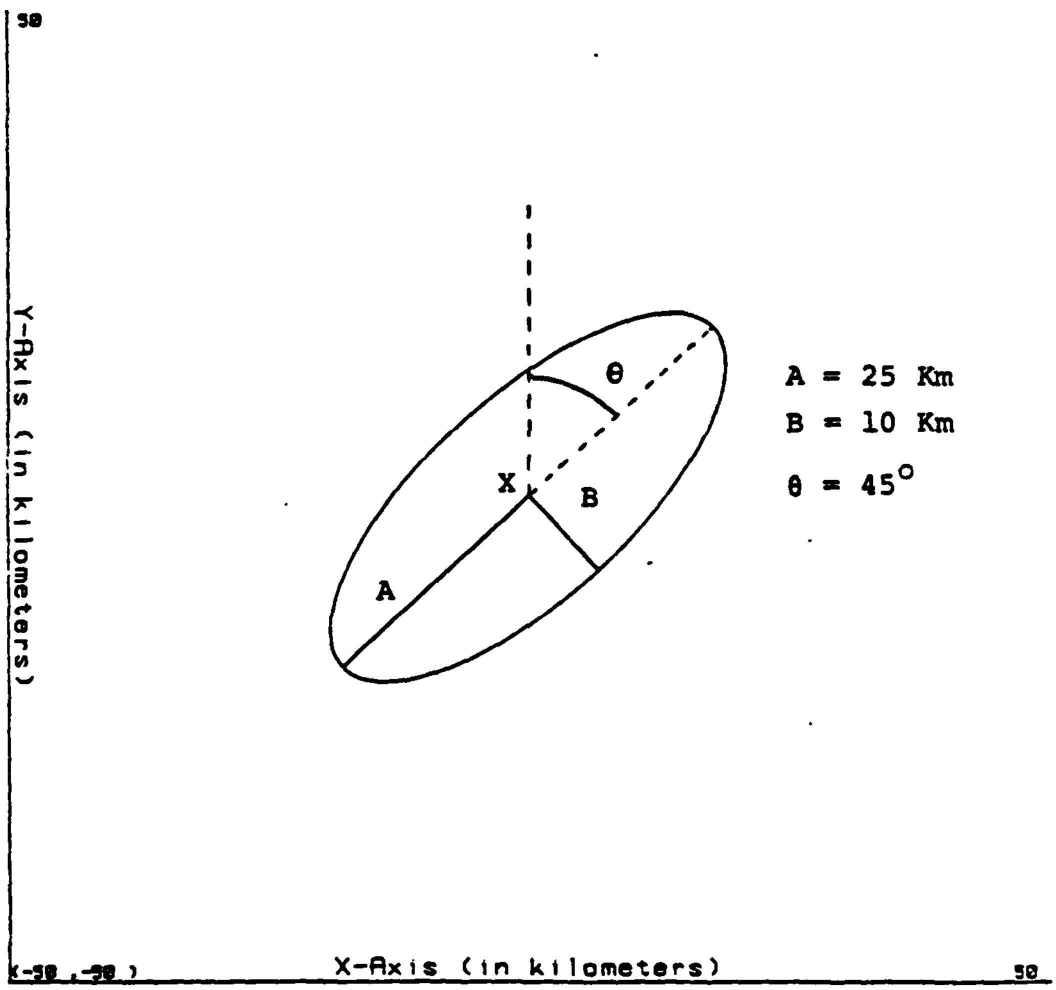
The purpose of the analysis center's data base is to store information about various intelligence objects. If two pieces of data come in, both of which concern one object, such as an emitter, it makes sense to pool the information in some way. Increased information should give a better estimate of the emitter's location. Self-correlation tests whether two pieces of data should be pooled, or combined, and if so performs the combination. Figure 2 displays a simple description of the self-correlation process. Note that this is not intended to describe any particular intelligence system, but is merely an explanation of the general concept.

A new location estimate and confidence ellipse is received by the analysis system. It is compared to each ellipse already in the data base. If the statistical test used by the system indicates that the new data refers to the same emitter as an ellipse in the data base, the new ellipse is combined with the old ellipse, and the combined data is stored. If the statistical test indicates that the new data do not refer to an emitter in the data base, the new data is entered as a separate entry.

Figure 3 shows two ellipses that the test rejects as being from the same emitter. Figure 4 shows two that the test accepts, as well as their combined, or resultant, ellipse.

The goal of this paper is to examine the geometric implications and properties of the statistical test for deciding whether two estimates refer to the same emitter, and of the method of combination for confidence ellipses.

FIGURE 1



A 95% Confidence Ellipse

FIGURE 2:

PATH OF NEW DATA THROUGH SELF-CORRELATION

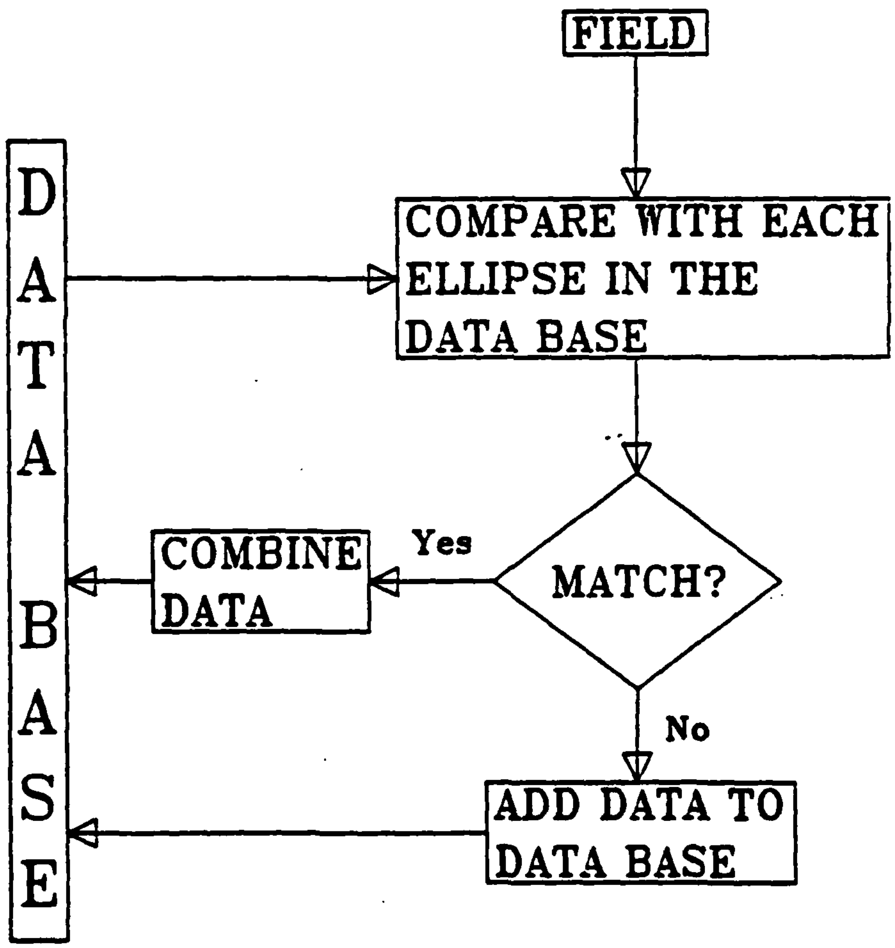
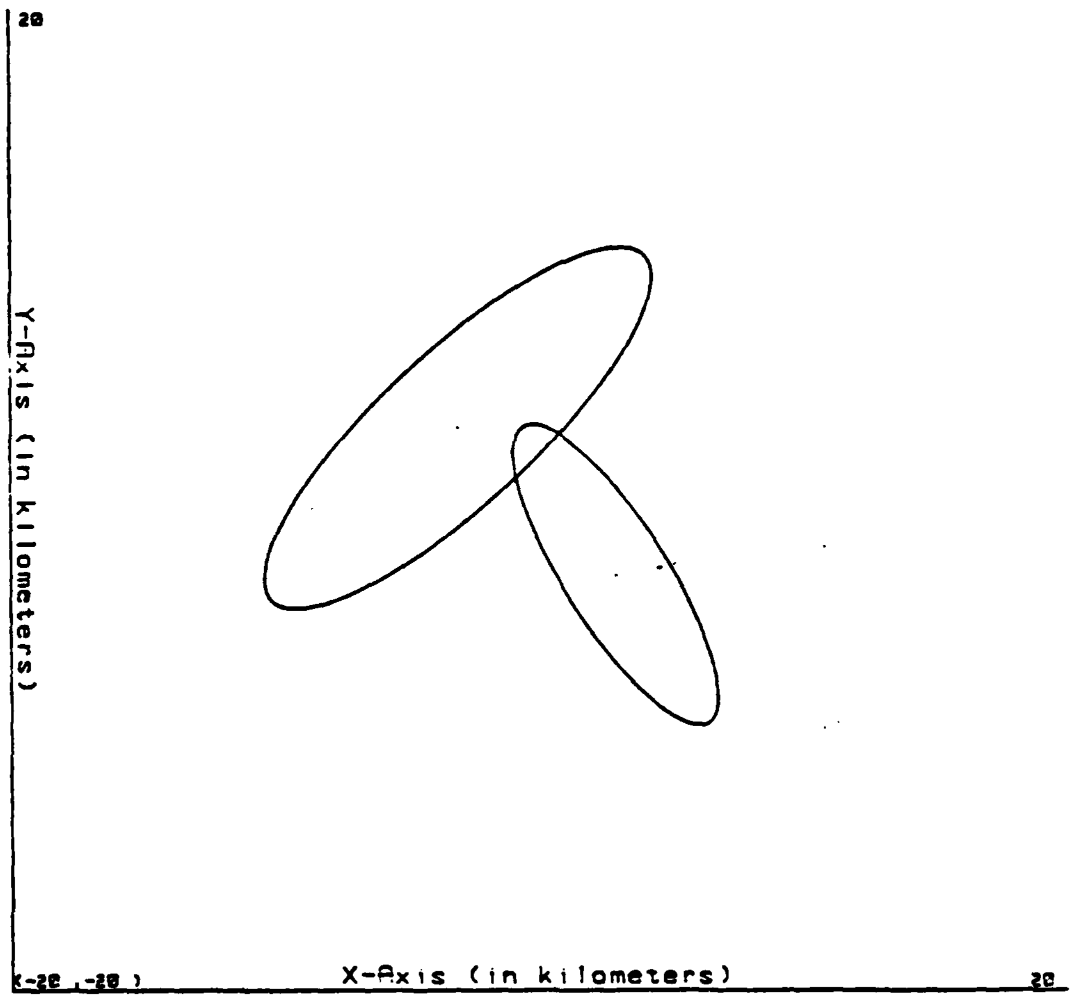
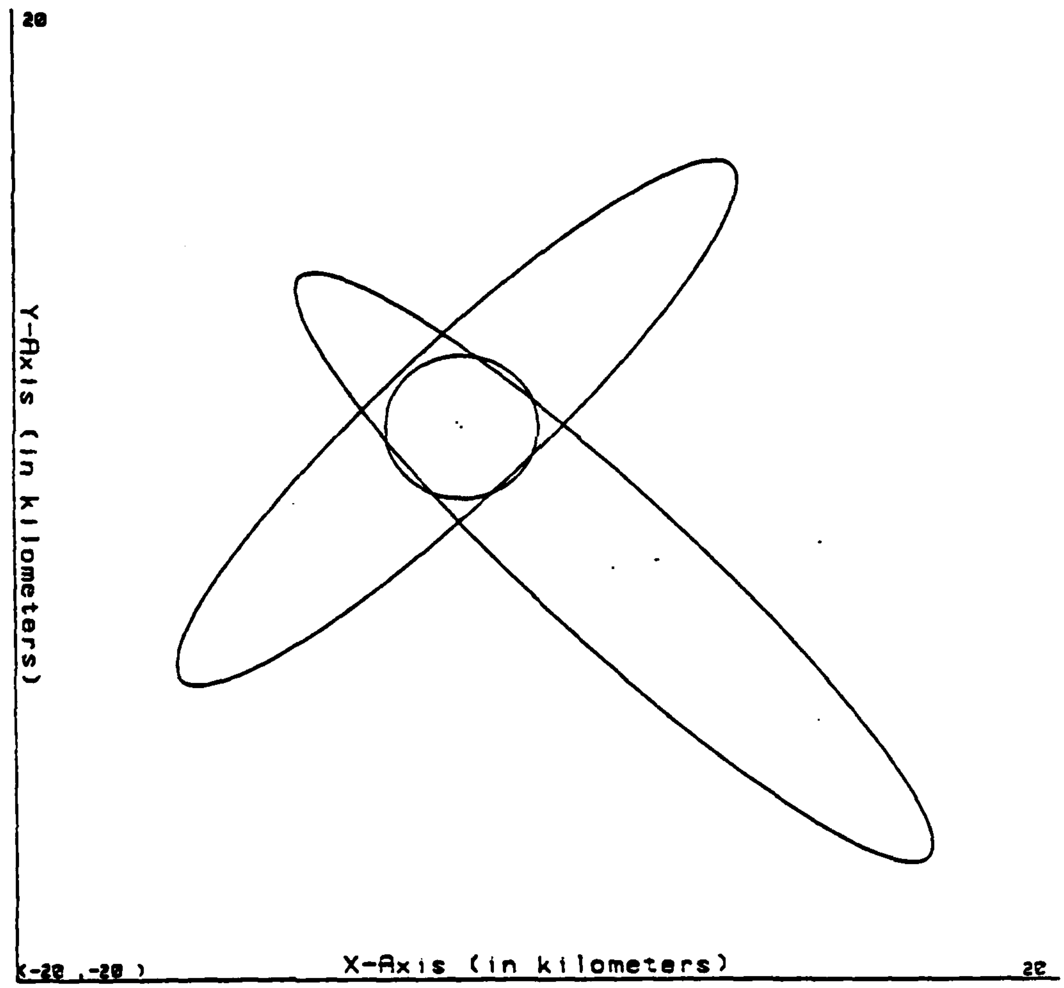


FIGURE 3



Two ellipses which <u>are not</u> allowed to combine

FIGURE 4



Two ellipses which <u>are</u> allowed to combine

First, understanding these properties provides an intuitive basis for looking at self-correlation. Will two ellipses be accepted as coming from the same emitter? How will they combine? Questions such as these may frequently be answered simply by looking at the two ellipses. Further, the test and combination method are based on a number of assumptions which may or may not be met in actual use. If one of these assumptions is not met, then the statistical test or the combination algorithm may give spurious results. The geometric properties described in this report provide a means of understanding how robust these methods are with respect to various assumptions -- that is, how large an effect relaxing an assumption will have. Lastly, the properties discussed in this report shed light on the kind of problems which will naturally arise while using the statistical test and combination method for self-correlation.

To this end, the following points will be discussed:

Shape, size, and orientation of the resultant ellipse.
Location of the resultant ellipse.
Geometrical properties of combining more than two ellipses.
Geometrical properties of the statistical test between two ellipses.
Properties of the statistical test used on a sequence of ellipses.

II. CONCLUSION

The geometrical properties of the statistical test for deciding if two estimates come from the same emitter and of the combination method for confidence ellipses will be discussed in sections III and IV. However, there are certain broad results which will be stated here, particularly as regards combining or testing a sequence of estimates.

If the combination method is used to combine a sequence of estimates, the order is not important; the final resultant ellipse will be unchanged. However, the statistical test is not independent of order.

Refer back to Figure 2, concerning the self-correlation algorithm. In Figures 5, 6, 7, and 8 we will trace a series of ellipses through self-correlation. Note that the ellipses in these figures (which will be called A, B, and C) are drawn to scale -- if the text says that the test will accept two ellipses as coming from the same emitter, this is the case for the two ellipses shown. Where two ellipses are combined, the resultant ellipse will be called G, H, or I.

Case 1: A, B, C.

Initially, there are no ellipses in the data base. Sensors in the field send ellipse A into the analysis center, where it goes through self-correlation. Since there are no other ellipses to compare A with, it is entered into the data base.

Now, ellipse B is sent in from the field, and goes through self-correlation. It is compared with A, and the statistical test accepts that A and B refer to the same emitter -- that is, they are both estimates of the location of a single emitter. B is therefore combined with A. A, B, and their combination is shown in Figure 5a.

Since A and B were combined, there is only one ellipse in the data base: the combination of A and B, which is called G. G is shown in Figure 5b.

A third ellipse is received from the field, ellipse C. It is compared with G, and the test accepts that they refer to the same emitter. C, G, and their combination is shown in Figure 6a. The combination, ellipse H, replaces ellipse G in the data base (Figure 6b).

Thus, when ellipses A, B, and C are received and tested in that order, the final result is a single ellipse, ellipse H.

Case 2: A, C, B

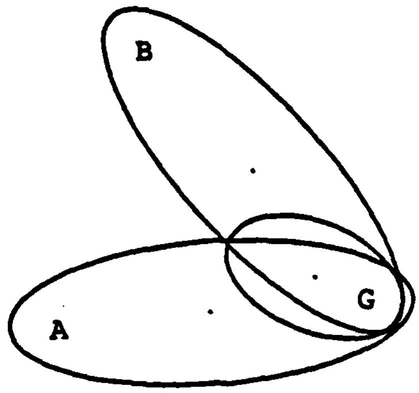
Now, suppose these ellipses were received in a different order. This could easily change the results.

As before, A is the first ellipse sent in from the field, and is thus the first ellipse in the data base. Next, ellipse C is received. The statistical test rejects that A and C refer to the same emitter, and so no combination is done. As indicated by Figure 2, ellipse C is entered into the data base. A and C are shown in Figure 7a.

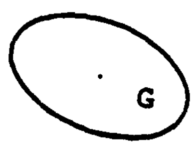
Next, ellipse B is received from the field, and is compared to both A and C (Figure 7b). The statistical test accepts that A and B refer to the same

FIGURE 5

5A

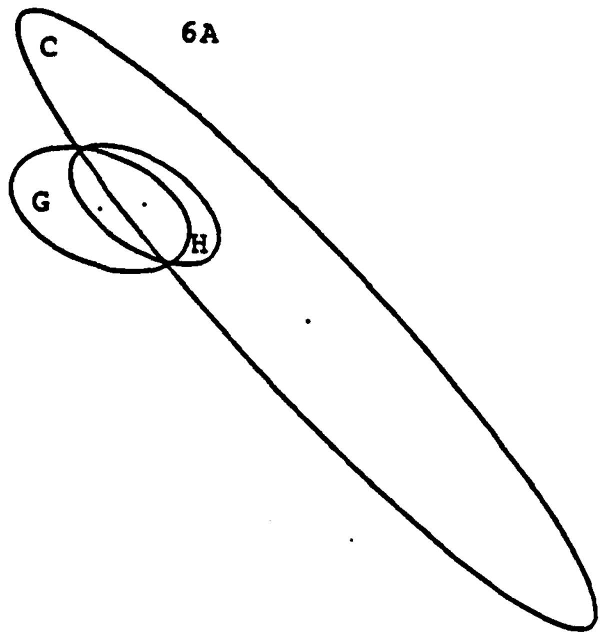


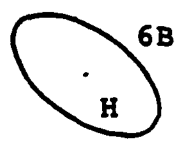
5B



Ellipses A and B are tested and combined

FIGURE 6

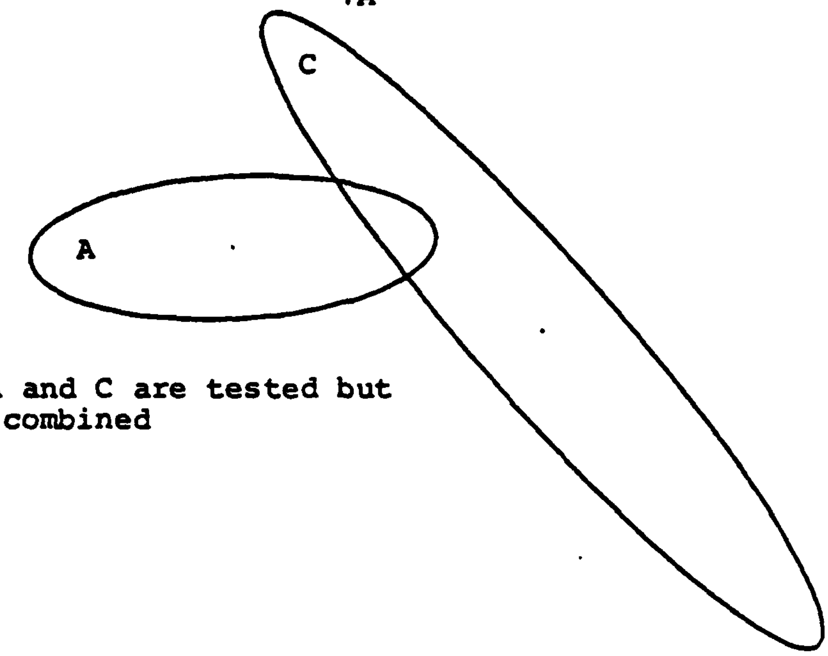




Ellipses C and G are tested and combined

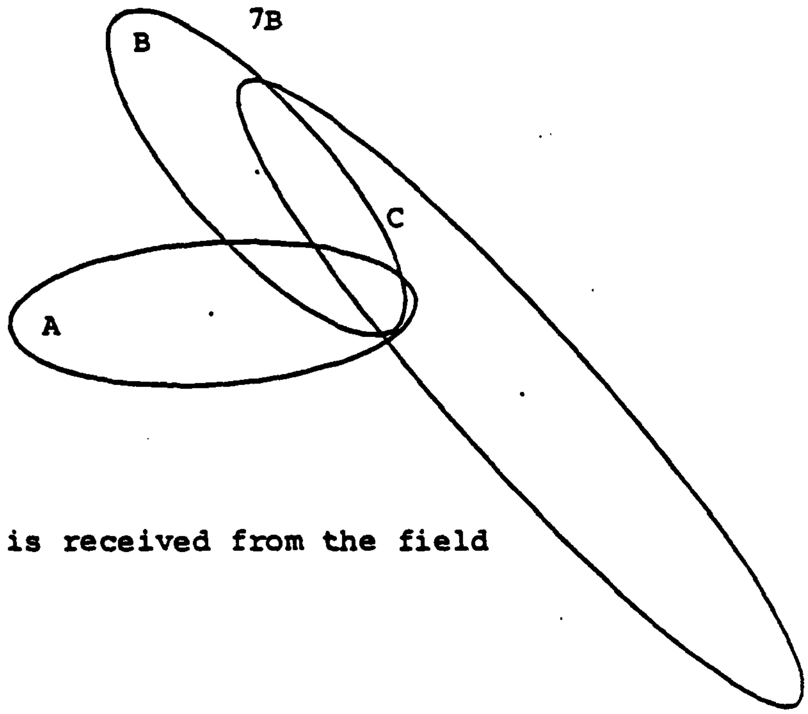
FIGURE 7

7A



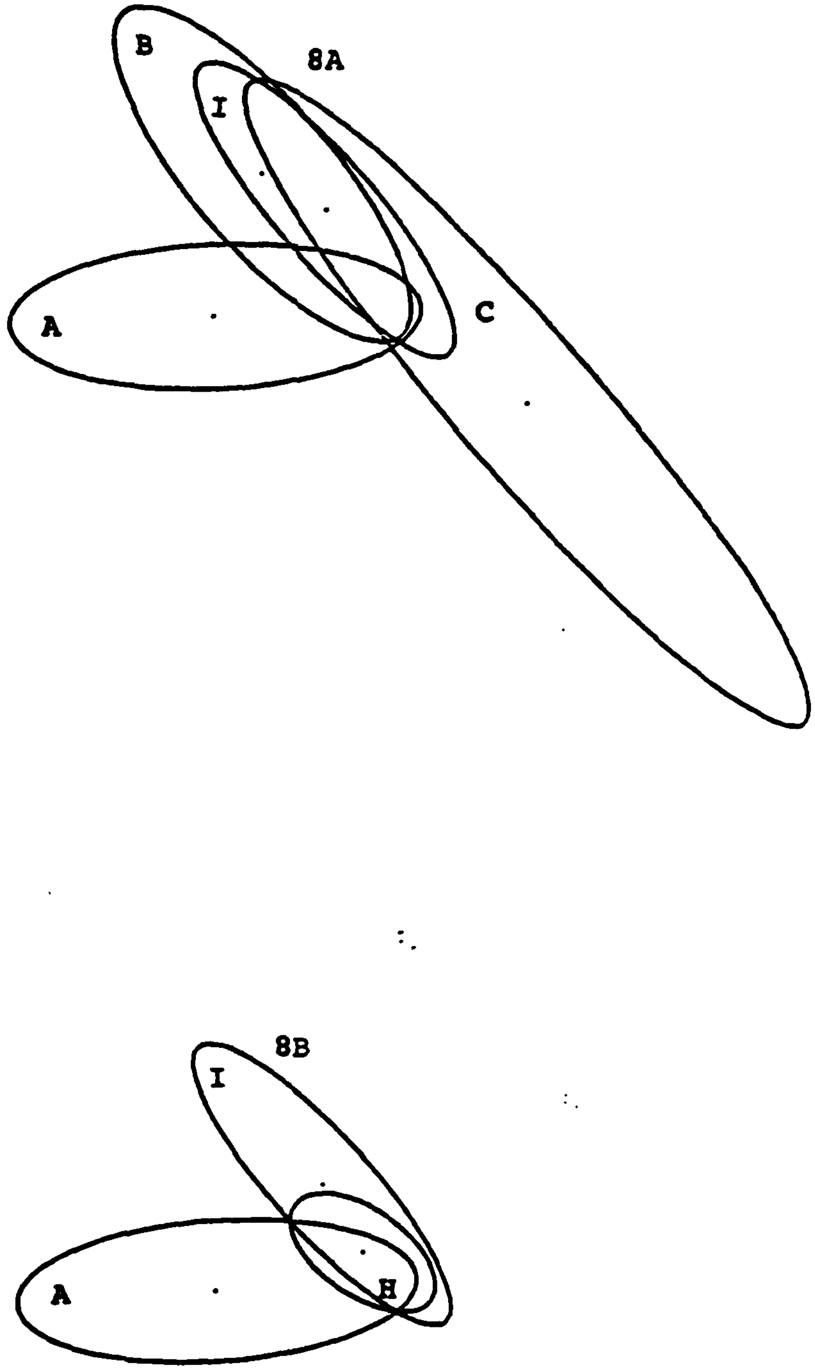
Ellipses A and C are tested but
cannot be combined

7B



Ellipse B is received from the field

FIGURE 8



Ellipses B and C are tested and combined

Ellipse H is the result of the other order of combination, provided for comparison.

emitter, and that C and B refer to the same emitter, even though it would not accept that A and C refer to the same emitter! It is unclear what should be done in such a case. B can not be combined with both A and C, and A and C cannot be combined with each other.

We assume, arbitrarily, that B is matched with ellipse C rather than A. Thus, B and C are combined, producing ellipse I, shown in Figure 8a.

Finally, ellipse I replaces B and C in the data base (Figure 8b).

When the ellipses are received and tested in the order A, C, B, the self-correlation data base ultimately contains two ellipse, A and I. Note here that if B had been combined with A instead of C, two ellipses still would have resulted. Figure 8b also shows ellipse H, the result of Case 1, for purposes of comparison.

It should be noted, however, that the example above is based only on one possible version of the self-correlation algorithm. Here, an ellipse estimate is only compared with the ellipses in the data base when it is first received from the field. A more sophisticated self-correlation process might also periodically test all of the ellipses in the data base with each other, combining where appropriate. If such a process had been used in the example above, then the results of the two cases above may have been somewhat different. The point to be made here is that the order in which the ellipses are tested and combined will <u>still</u> make a difference in the final results.

Thus, while the order is not important to the combination method in and of itself, the order of the ellipses can have a tremendous impact upon the statistical test, and thus on the self-correlation algorithm as a whole. This is perhaps the most important result in this paper.

III. GEOMETRIC PROPERTIES OF THE COMBINATION METHOD

In this section, a number of properties of the combination method are listed. The figures are at the end of the section. Mathematical details and proofs are contained in the Mathematical Appendix. Note that all of these results are dependent upon the ellipses being constructed at the same level of confidence (i.e. 95%, 90%, etc.).

A. The resultant estimate of target location is dependent upon two original point estimates, and <u>also</u> upon the shape, size, and orientation of the original ellipse estimates.

Comment: Errors in ellipse shape and size can affect target location estimates.

B. The size, shape, and orientation of the resultant ellipse is dependent only on the size, shape, and orientation of the two original ellipses, and not on their relative positions. (See Figures 9, 10, and 13).

Comment: This implies that the <u>distance</u> between two estimates of target location does not affect the algorithm's 'certainty' of the resultant estimate of target location. This assumes that we are correct in accepting that the two estimates are from the same emitter.

C. If the original point estimates are equal, the resultant point estimate will be equal to them. Stated equivalently, the resultant confidence ellipse will be centered at the same point as the original ellipses. (See Figure 11).

D. If the original point estimates are equal, that is, the original ellipses are centered at the same point, then the resultant ellipse will be wholly contained within their intersection. (See Figures 11 and 12).

Figure 13 demonstrates both point B and point D. In 13a, the resultant ellipse is contained within the intersection of the original ellipses. In 13b the positions (point estimates) of the original ellipses have been moved, but the resultant ellipse has the same size, shape, and orientation as in 13a.

E. When the original point estimates are not equal, the resultant ellipse will be approximately positioned upon the intersection of the original ellipses. (See Figure 14).

Comment: This is not the case, of course, when the ellipses do not intersect. However, the statistical test will not accept two ellipses for combination if they do not intersect, so this

qualification is irrelevant. (See Section IV, E).

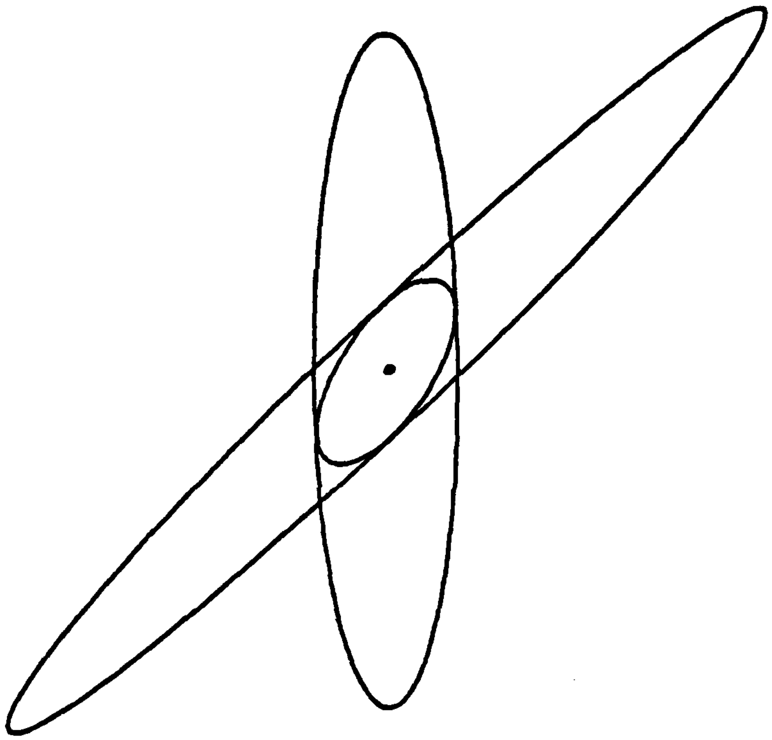
F. If the original ellipses are tangent, but not otherwise intersecting, the point estimate of the resultant ellipse will be located near the point of tangency. Further, if the sizes of the original ellipses are increased by a factor of the square root of 2 in such a case, the intersection of the expanded ellipses will contain the point estimate of the resultant ellipse. (See Figure 15, 16, 17, and 18).

Comment: This result is interesting mainly in light of point G, immediately below, since the statistical test will not accept two tangent ellipses for combination. (See point E of section IV).

G. If the two ellipses are not tangent, the resultant point estimate may be located geometrically as follows: decrease (or increase) the size of both original ellipses proportionally until they are tangent. Then the point estimate may be located as in point F above. In practice the size of the ellipses would only be decreased, since the statistical test will not accept two ellipses which do not intersect. (See point E of section IV.)

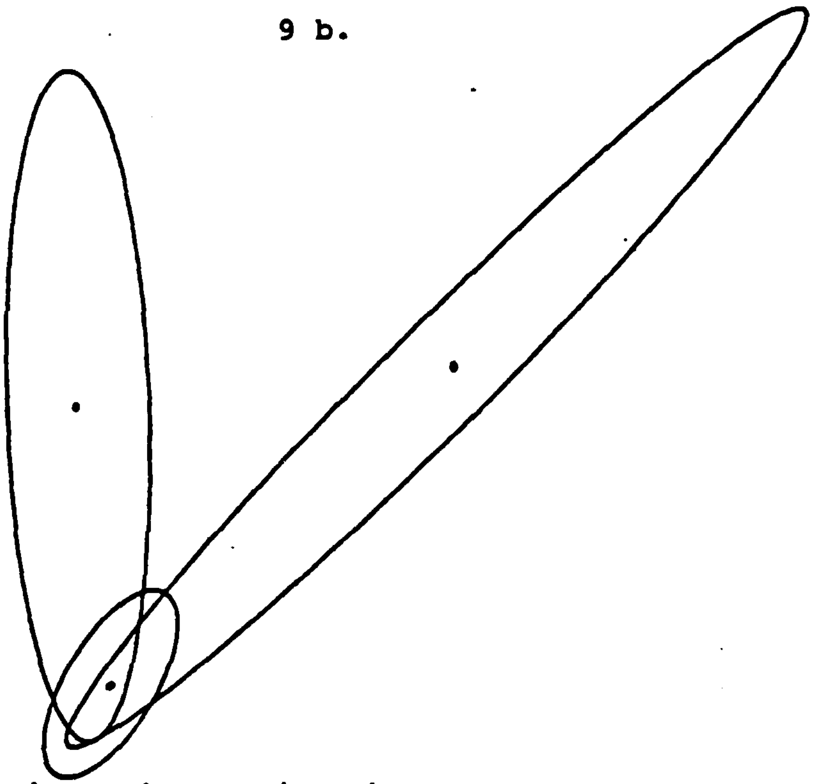
H. When more than two ellipses are combined the order of combination does not change the result. Thus the only difference order of combination can make is when the statistical test is used (to accept or reject that data are from the same emitter). (See Figures 19 and 20).

9 a.



This figure shows that a common center implies that the resultant ellipse is inside and similar to the intersection.

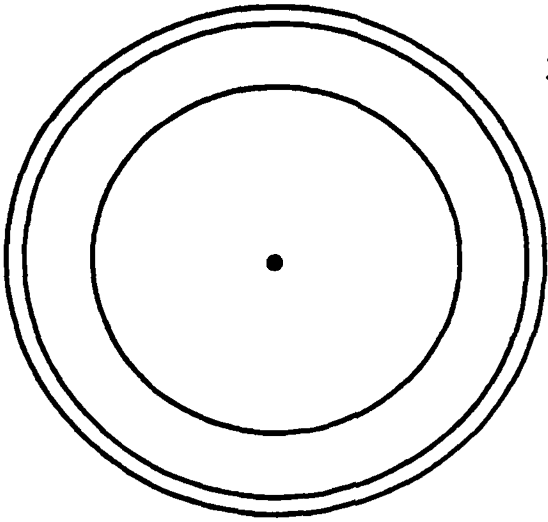
9 b.



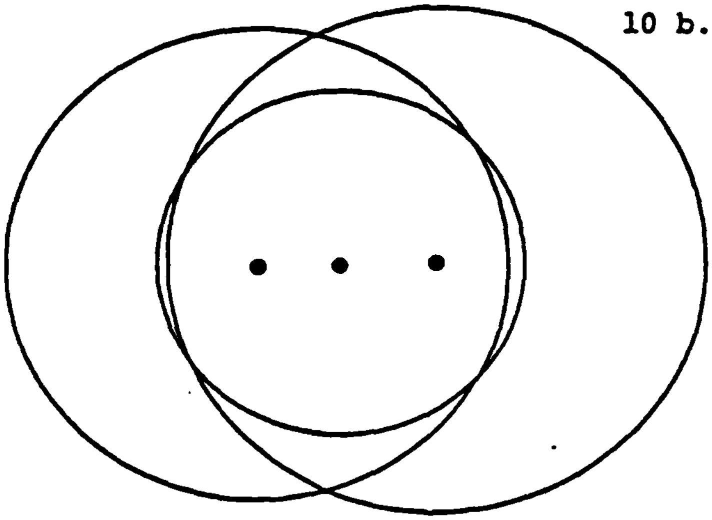
This figure shows that moving the centers apart can change the intersection rule. It also shows that the resultant ellipse is the same shape as it was in 9a.

Figure 9

10 a.

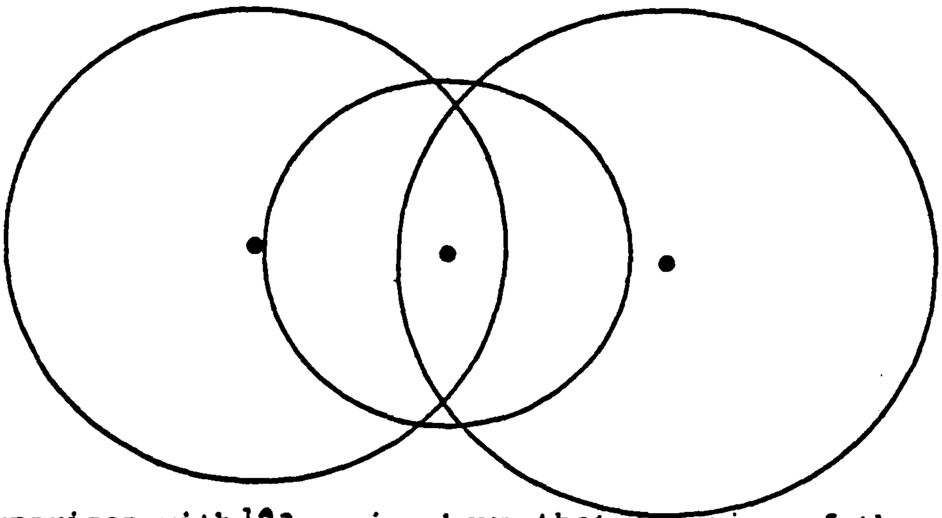


10 b.



Comparison with 10a again shows that location of the centers does not affect shape of the resultant ellipse.

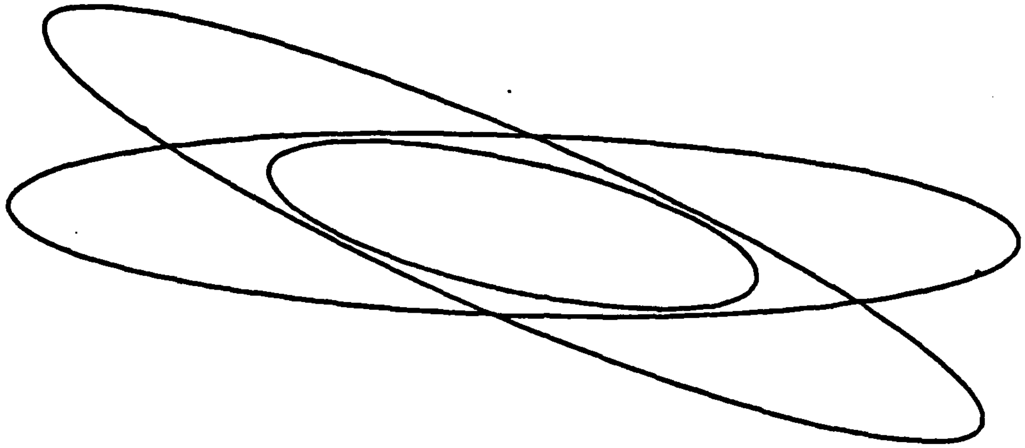
10 c.



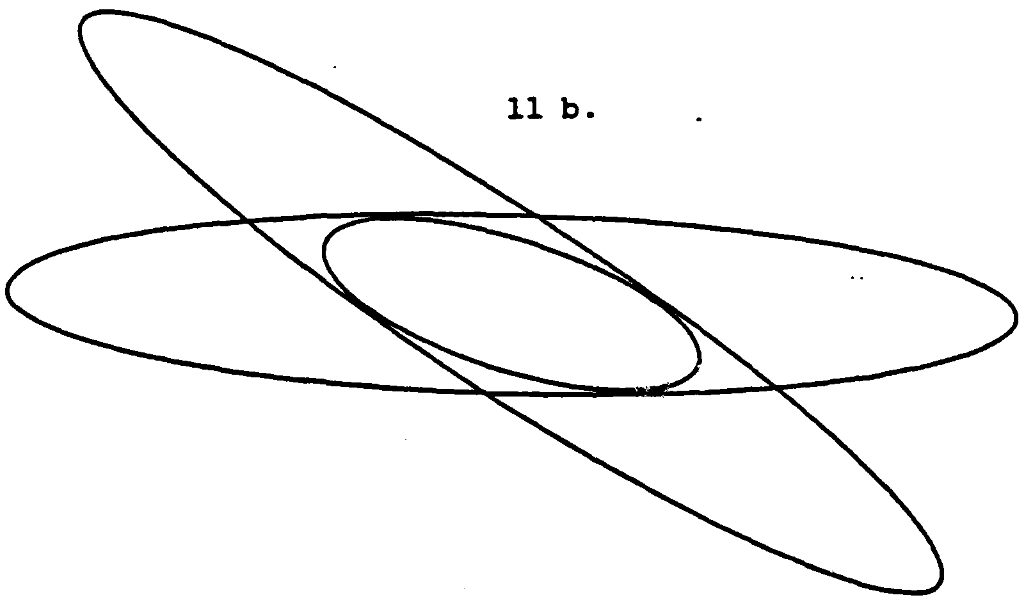
Comparison with 10a again shows that location of the centers does not affect shape.

Figure 10

11 a.

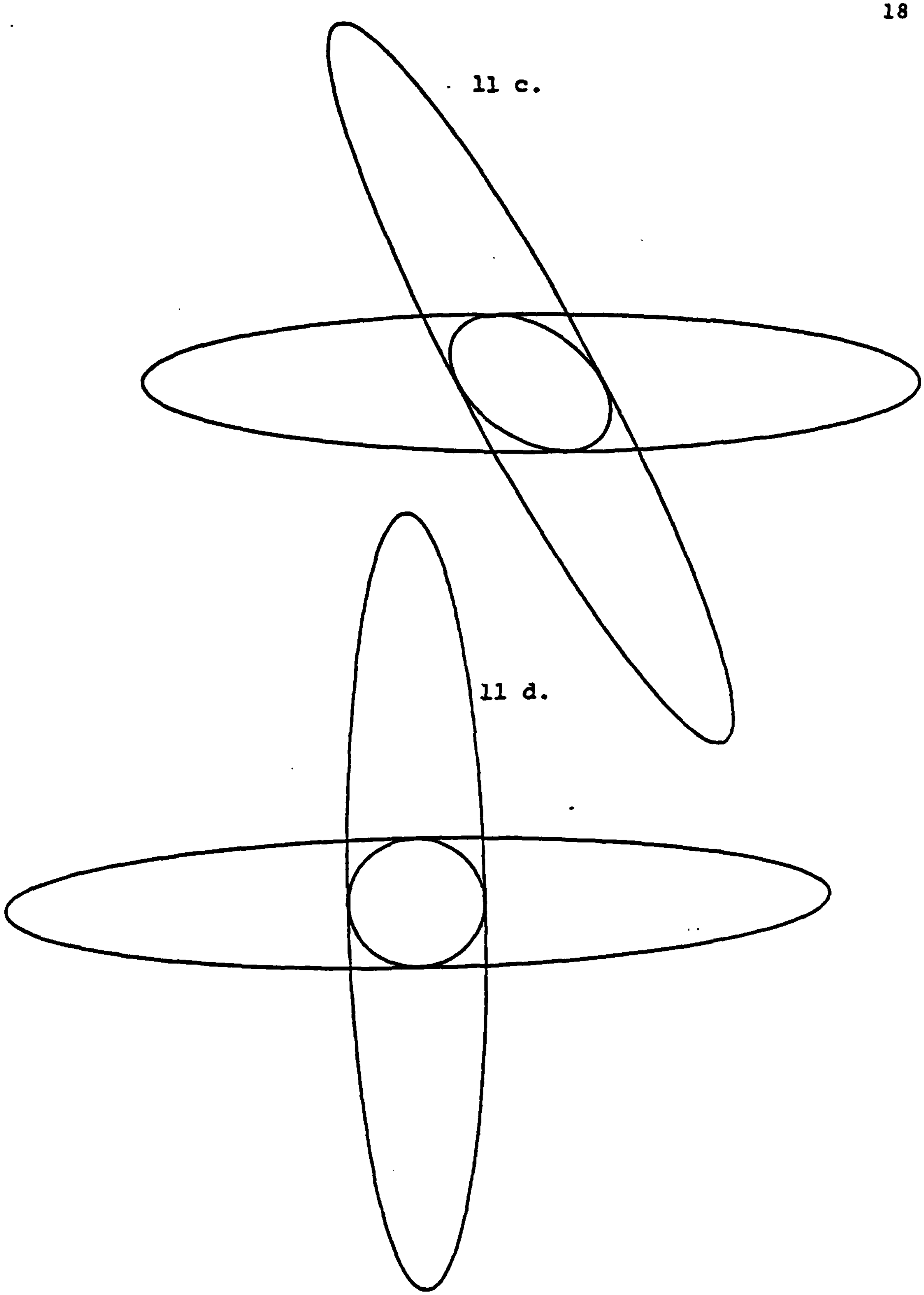


11 b.



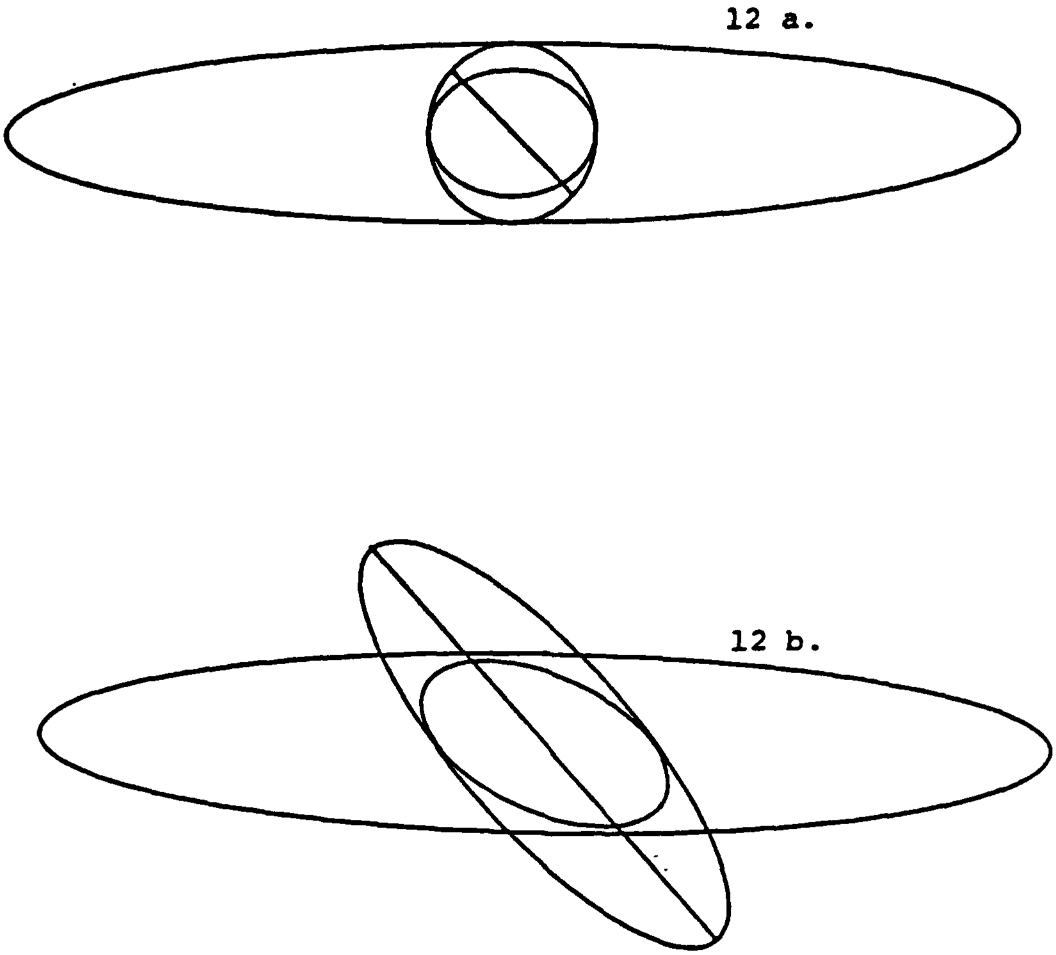
More examples with a common center. Note that the ellipse *angle was changed* slightly, significantly affecting the size of the resultant ellipse.

Figure 11



Compare with the previous page.

Figure 11 (Con't.)



Note that only the length of the axis of one of the ellipses is changed. The resultant ellipse changes only slightly from 12b to 12c.

Figure 12

12 c.

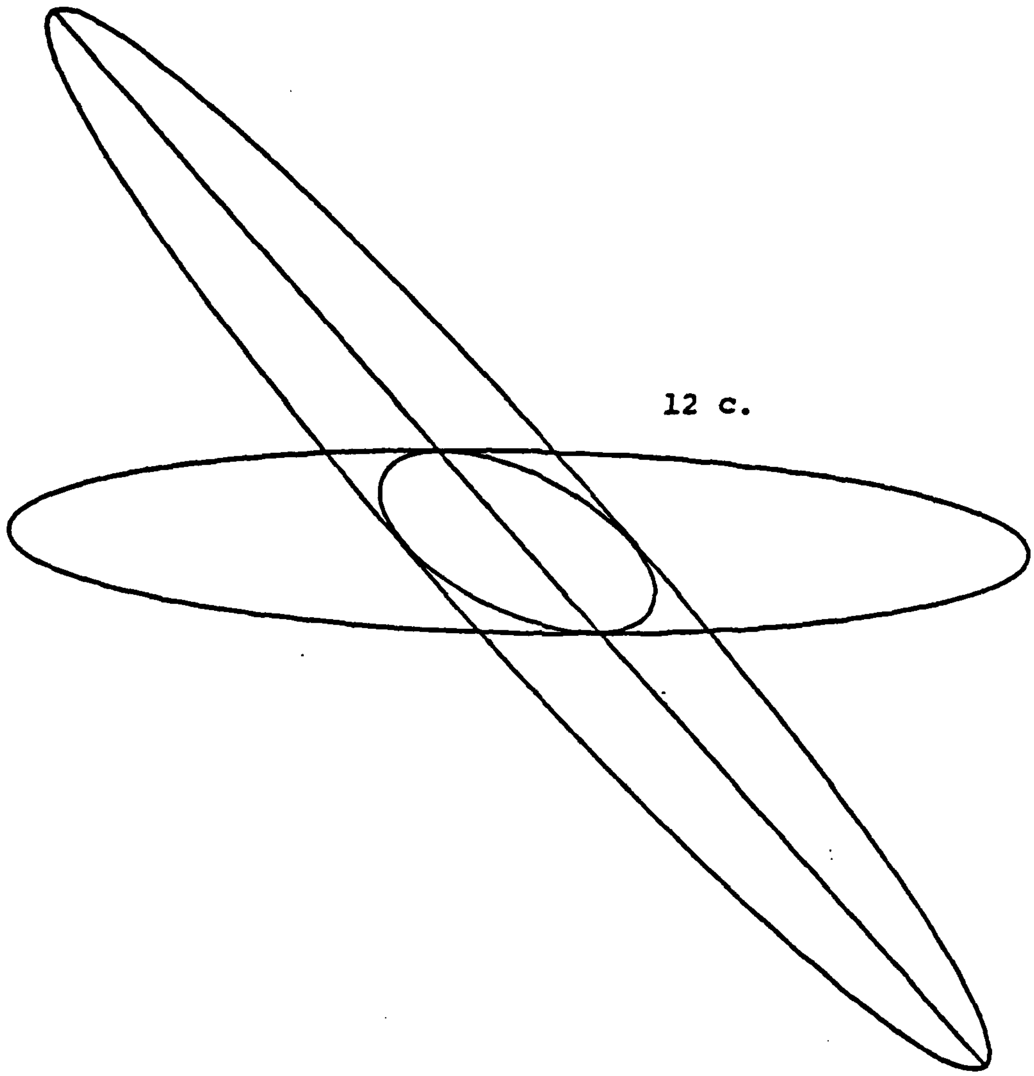
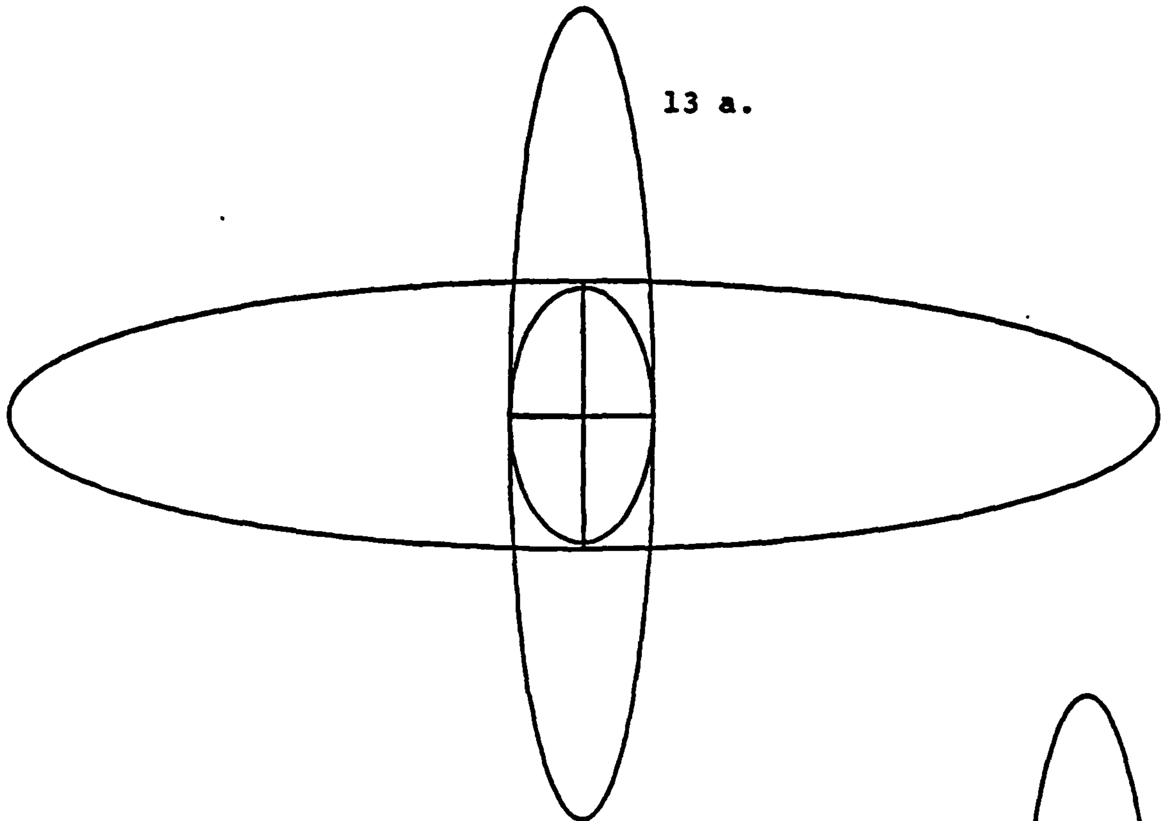
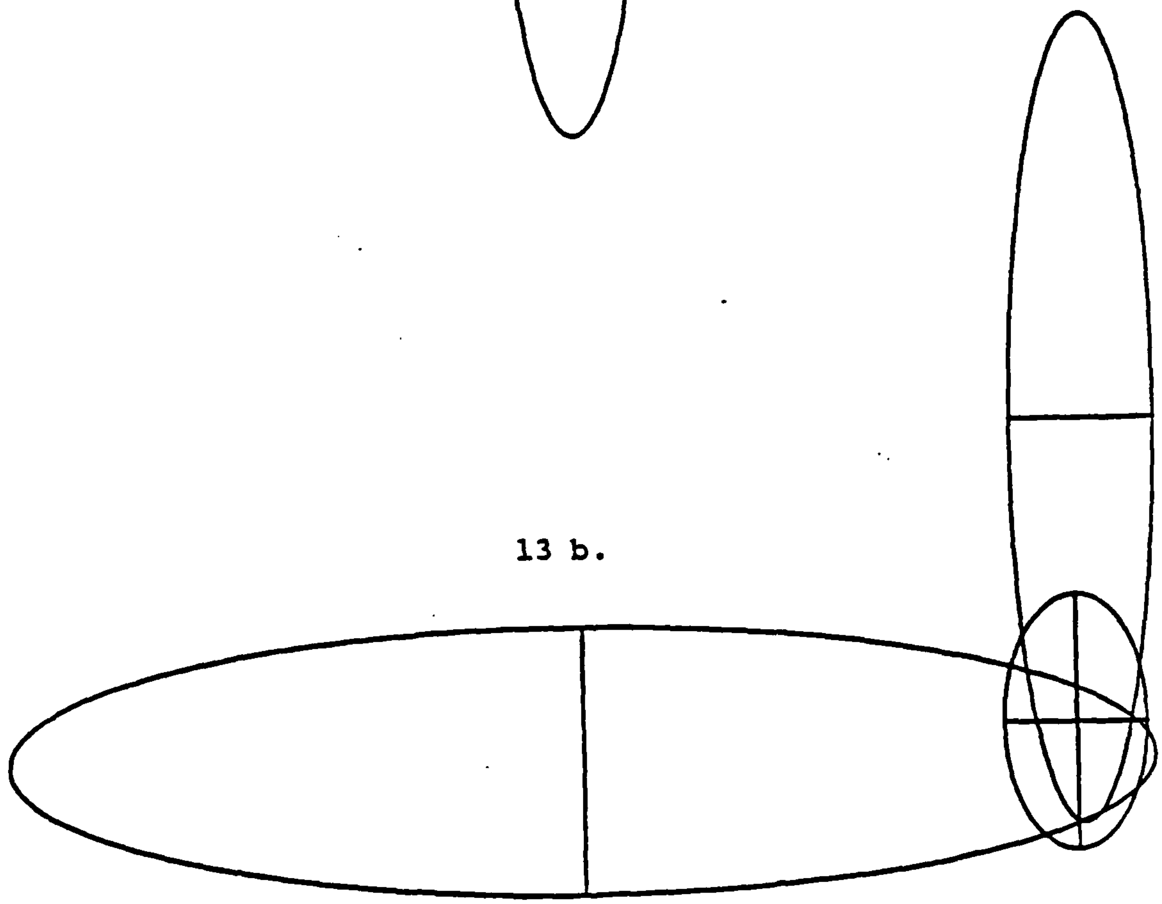


Figure 12 (Con't.)

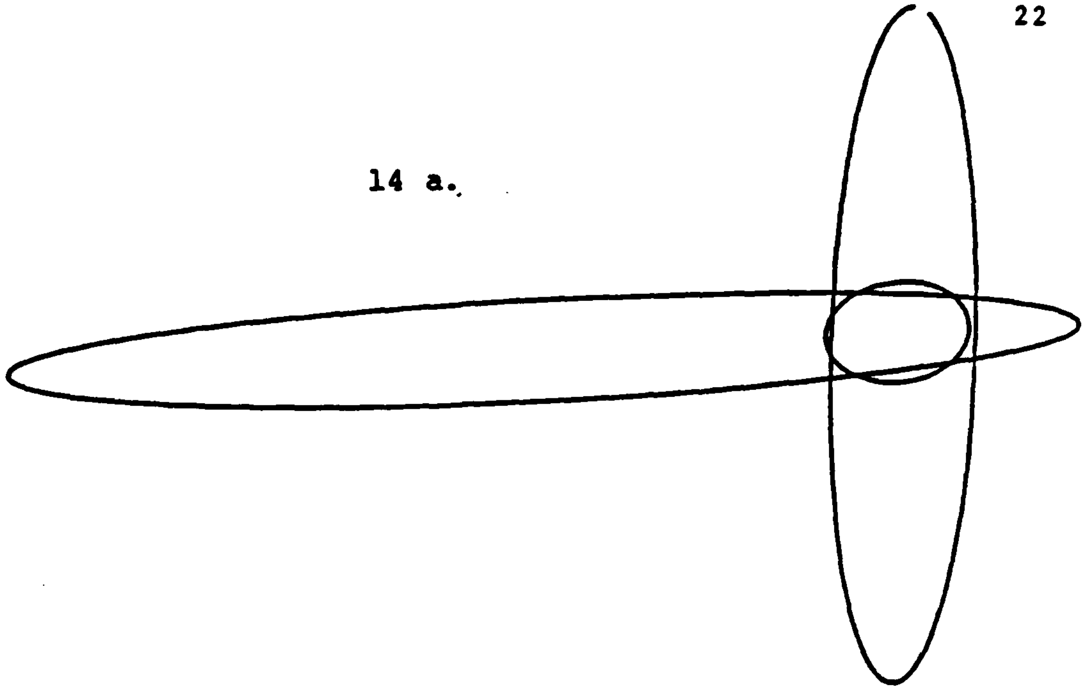




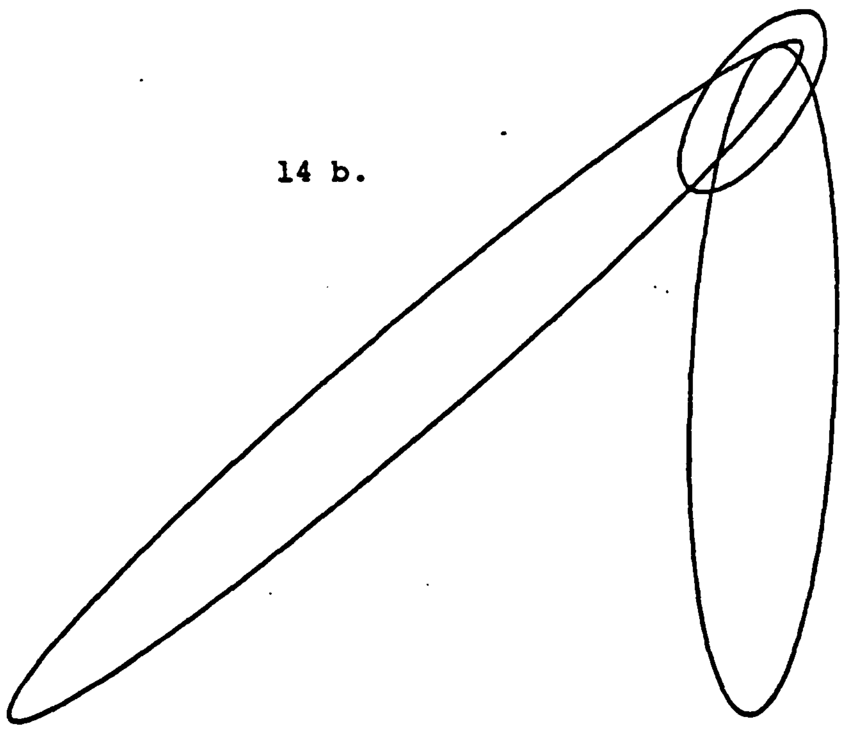
The resultant ellipse shape does not depend on the location of the original ellipses.

Figure 13

14 a.

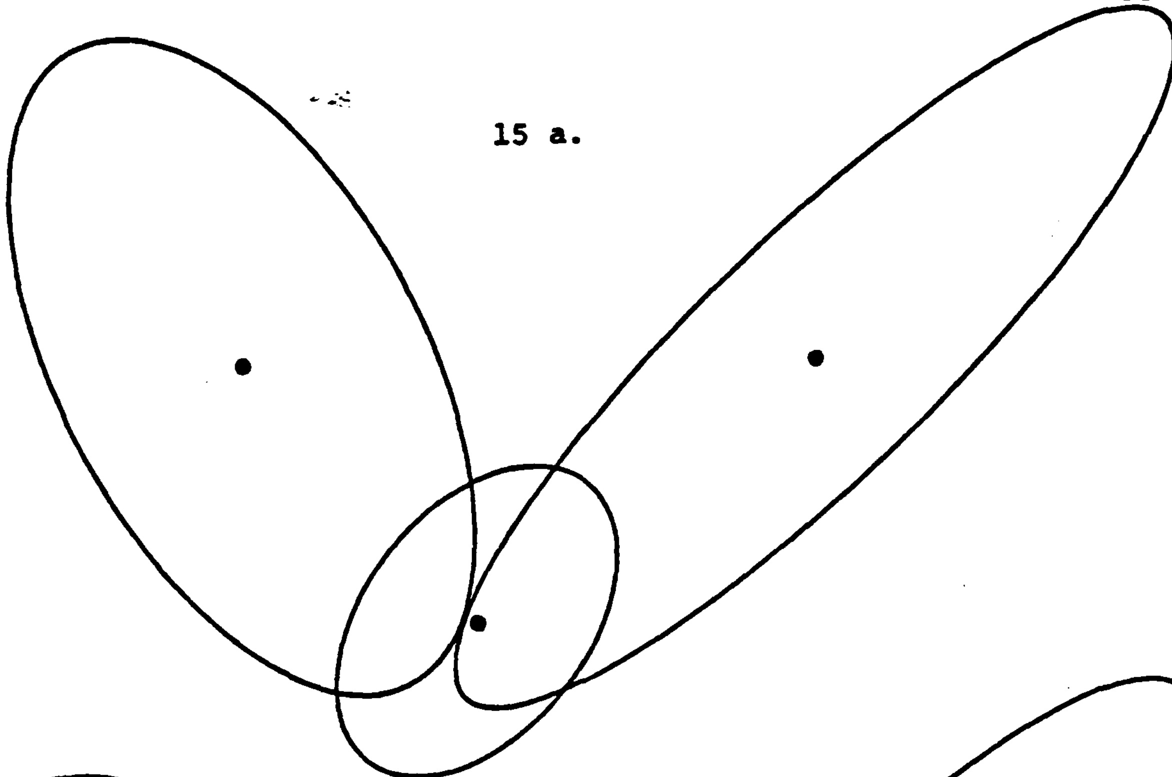


14 b.

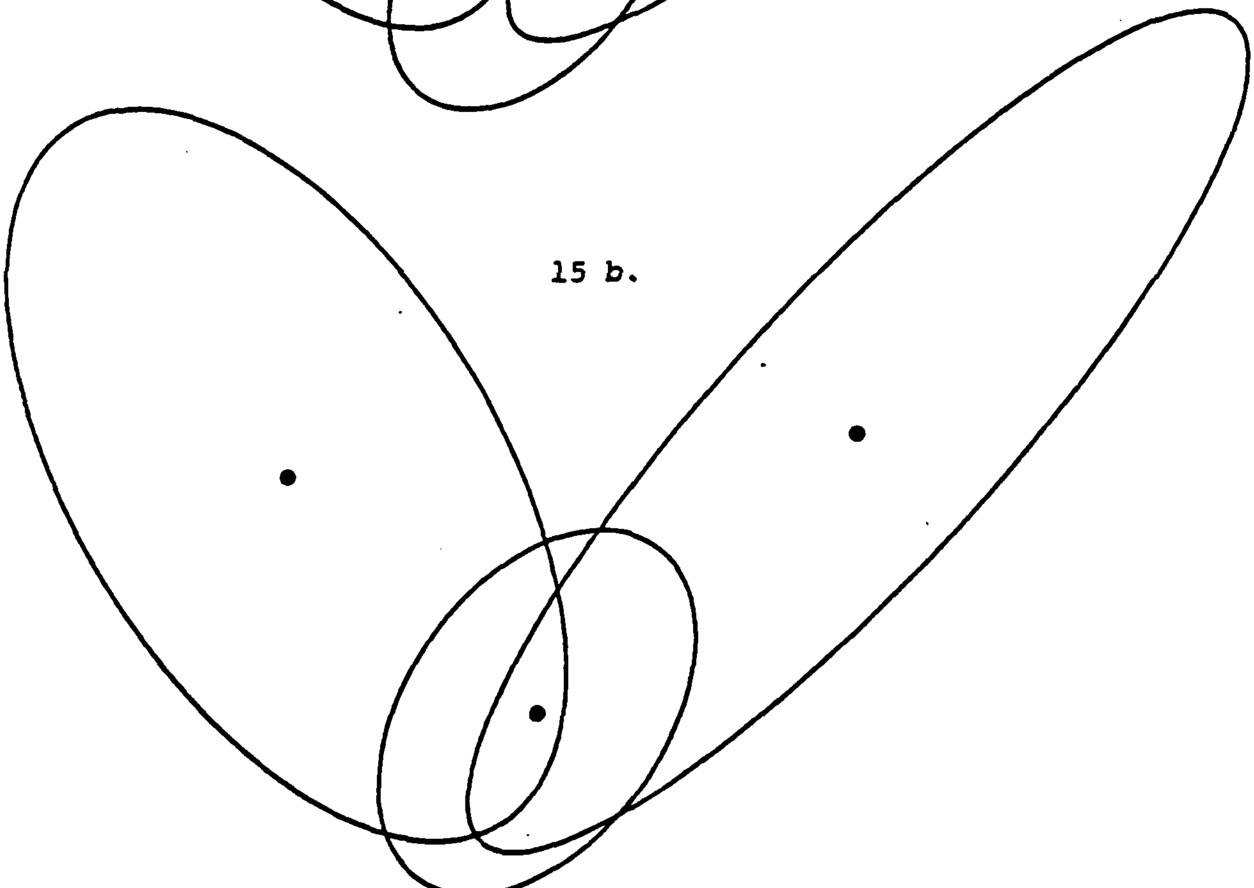


The first approximation to the location of the resultant ellipse is the location of the intersection.

Figure 14



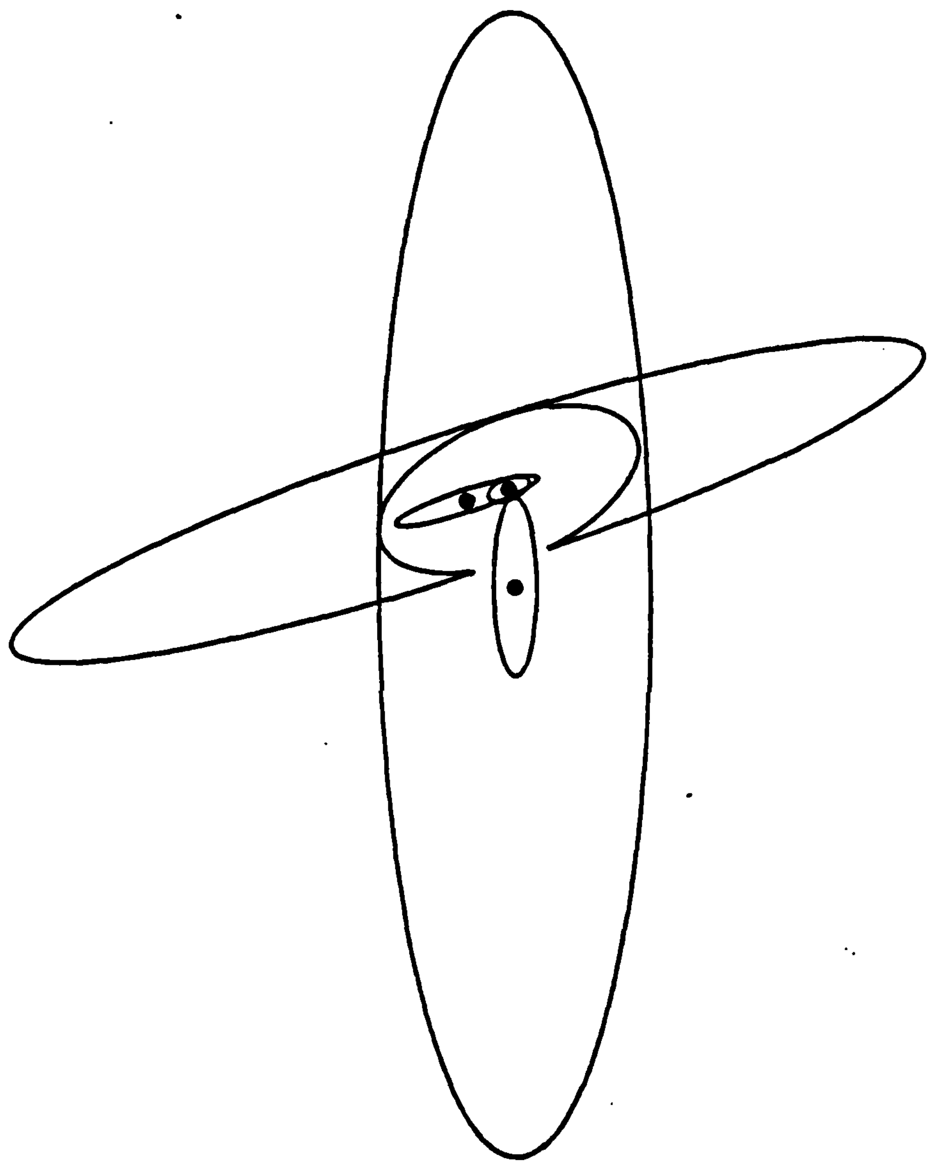
15 a.



15 b.

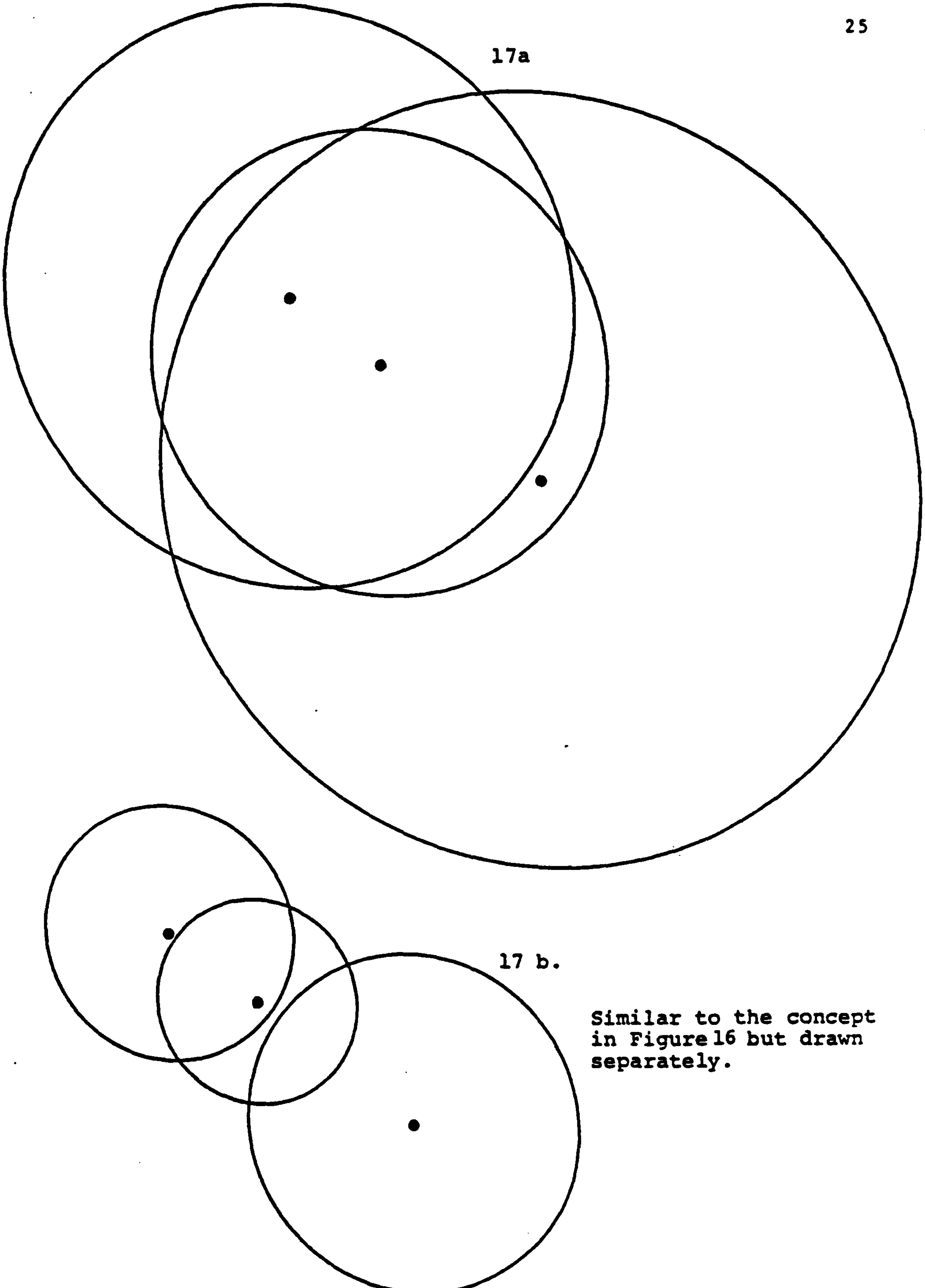
In 15a we see that the resultant location estimate is close to the tangency point. Figure 15b is $\sqrt{2}$ larger and the location estimate is inside the intersection as it will always be.

Figure 15



See Figure 15a for the basic idea. The difference here is that the original ellipses are not tangent but if they are shrunk proportionately, the point of tangency is close to the location estimate.

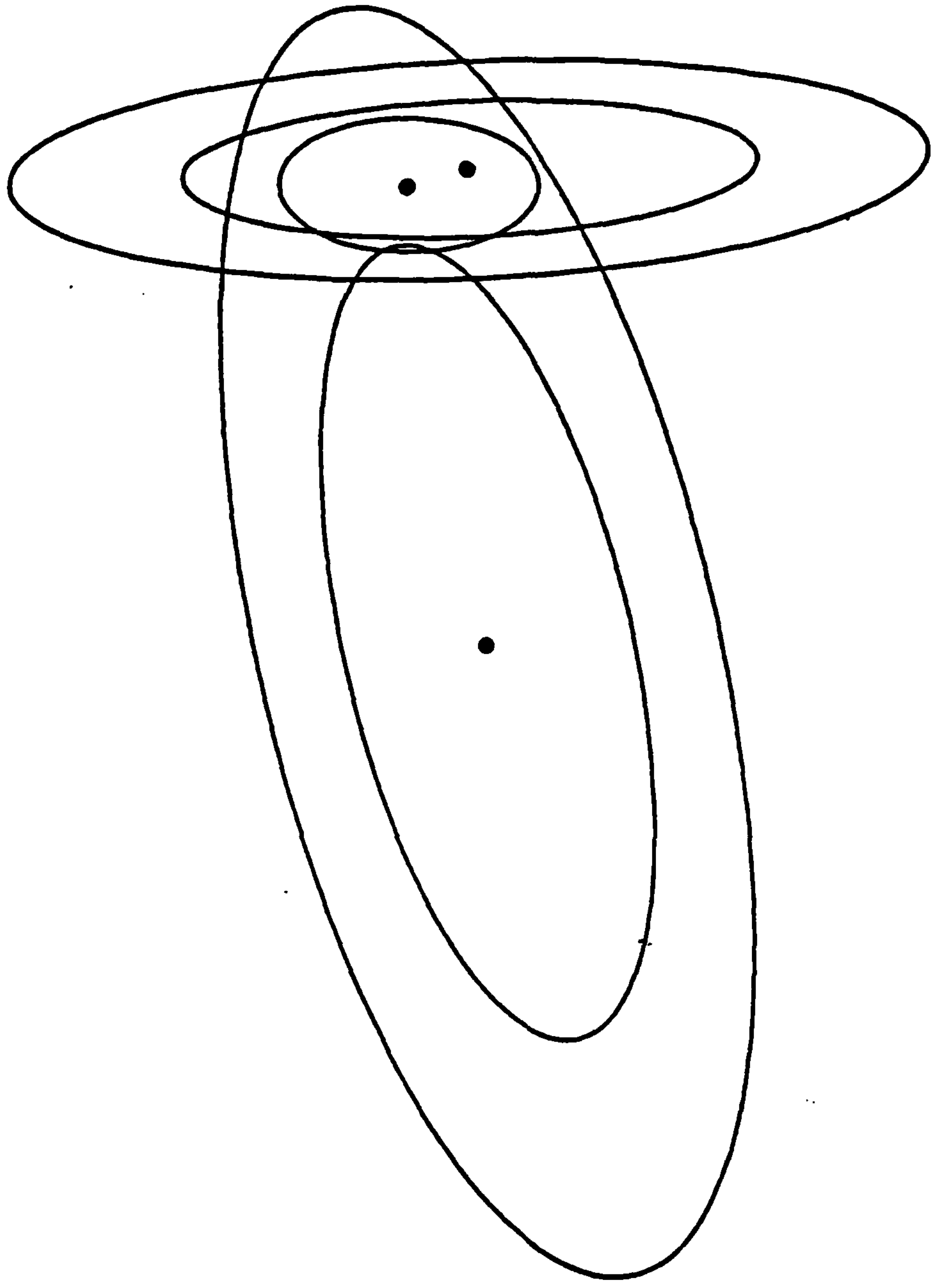
Figure 16



Similar to the concept in Figure 16 but drawn separately.

Figure 17

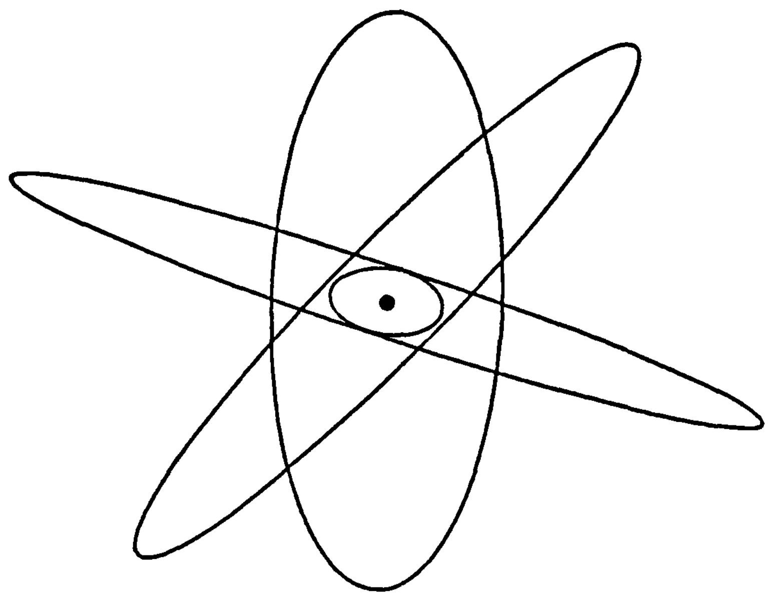
18



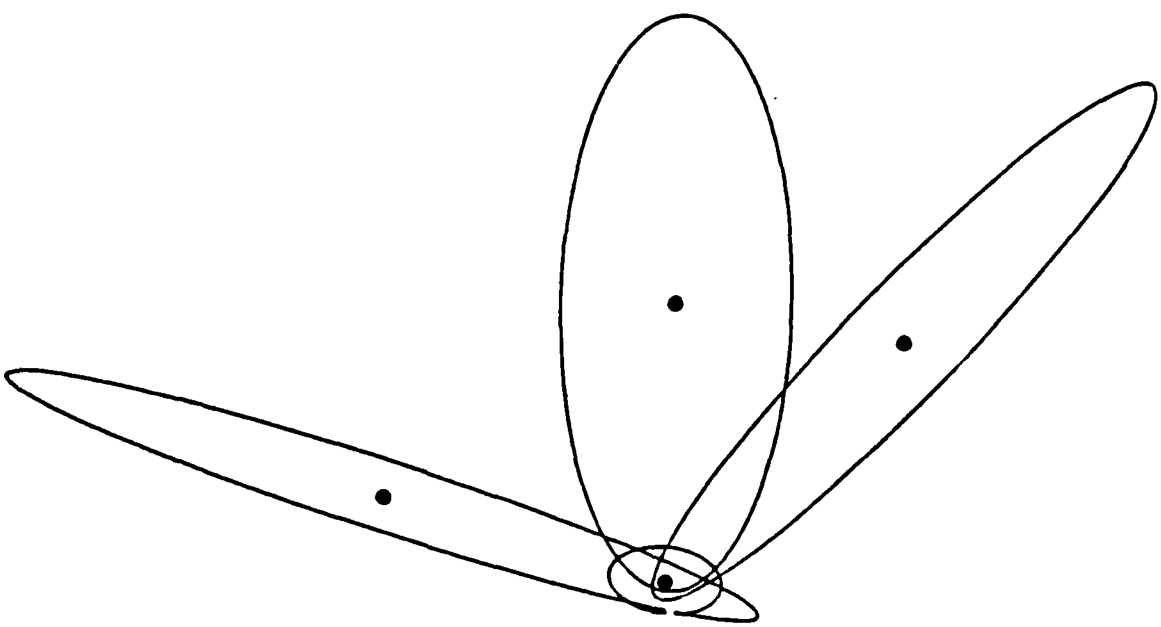
Similar to the concept in Figure 16

Figure 18

19 a.



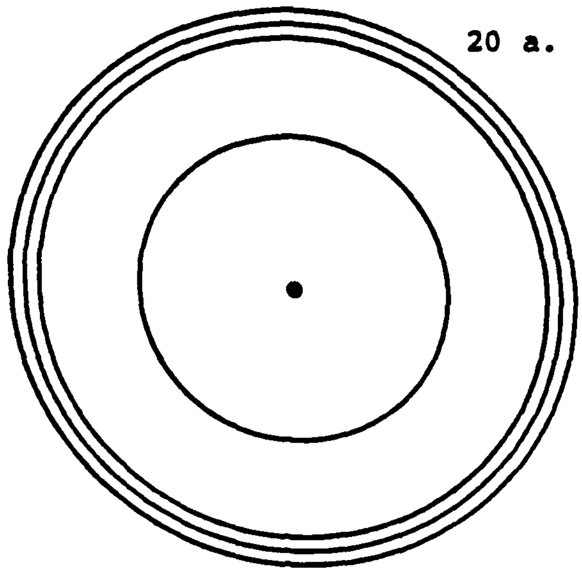
19 b.



The concept previously discussed for *three* ellipses. (The intersection property when there is a common center and one without a common center.)

Figure 19

20 a.



20 b.

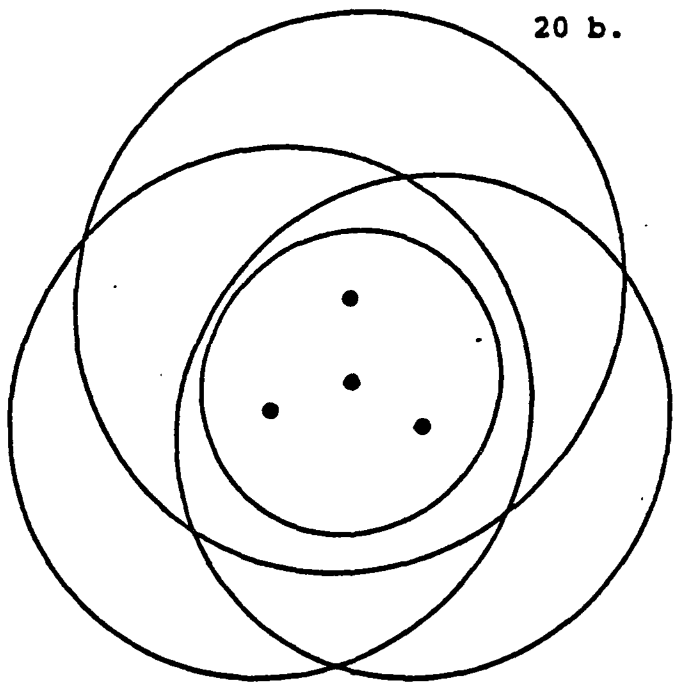


Figure 20

IV. GEOMETRICAL PROPERTIES OF THE STATISTICAL TEST

The following geometrical properties are proven in the Mathematical Appendix. Figures are at the end of the section. Note that these results assume that the statistical test's level of significance and the ellipses' level of confidence are equal (i.e. if the test is a 95% test, then all of the ellipses must be 95% ellipses).

A. The absolute orientations of the two original ellipses have no effect on the statistical test, only their orientations with respect to each other.

Figure 21a shows two ellipses, and Figure 21b shows the same two ellipses rotated an arbitrary number of degrees. The test result is the same in either case.

B. The units of measure have no effect on the test. One axis could be measured in meters and the other in kilometers.

C. Order of combination can matter when three or more estimates are involved.

This is an extremely important result. An excellent example is contained in the Conclusion (section II). Also see the "worst case" result, below.

D. If either of the ellipses in the test contains the center (point estimate) of the other ellipse, the test will definitely accept.

Figure 22 shows several cases of this result. The statistical test would accept in each case.

E. If the ellipses do not intersect at all, or only at one point (tangency), then the test will definitely reject.

Figure 23 shows several cases of this result. The statistical test would reject in each case.

F. There are cases which do not fit into either D or E above. For these cases, more complex geometric constructions are necessary to determine whether the test will accept or reject.

Figure 24 shows several cases in which neither point D nor point E can determine acceptance or rejection. 24a and 24b show cases in which the test would accept, and 24c and 24d show cases in which the test would reject.

The "Worst Case" Result of the Impact of Testing Order

As stated in point C above, the order of testing the ellipses can have a significant impact. This result shows the worst possible errors that can occur for three ellipses. First, however, we must define two pieces of notation.

Assume that we have three ellipses, A, B, and C. In order to test whether or not two of these ellipses refer to the same sensor, a test statistic is calculated. The exact formula of this statistic is shown in the Mathematical Appendix, but here we shall denote it by T(). For example, if A and B are being compared, the value of the test statistic would be T(A,B).

In order to test, we choose a value k which will give us the desired significance level. If $T(A,B) \leq k$, then we accept that A and B refer to a single emitter, otherwise we reject.

Suppose that $T(A,B) \leq k$ when the test is performed, and so the two ellipses may be combined. We will denote their combination by $A \oplus B$.

Three further observations must be made. First of all, if $T(x,y)=0$ for any two ellipses x and y, then x and y must be located at exactly the same point. Secondly, the value of $T(x,y)$ indicates in a statistical sense the distance between the locations of ellipses x and y. $T(x,y)=k$ is the largest value of $T(x,y)$ that will allow the statistical test to accept x and y for combination. Finally, if $T(x,y)=2k$, it is clear that the test will not accept x and y for combination.

We will show two possible "worst" results, Case 1 and Case 2. Both of these cases show what may happen when A, B, and C are tested in different orders. These results are dependent upon a mathematical result which is proved in the mathematical appendix, which says that no matter what order the ellipses are tested in, the sum of the test statistics is constant.

Case 1:

The shape and location of A, B, and C can be chosen so that the following is true:

Test Results	Order of Testing/Combination A, B, C	A, C, B
1st Test	$T(A,B) = k$	$T(A,C) = 2k$
2nd Test	$T(A \oplus B,C) = k$	$T(A \oplus C, B) = 0$

These statements state that when the ellipses are tested in the order A, B, C, the test will accept each time. The end result is the ellipse $A \oplus B \oplus C$. However, if testing is done in the order A, C, B, the test will reject ellipses A and C for combination, even though $A \oplus C$ and B must have the same location ($T(A \oplus C) = 0$).

Case 2:

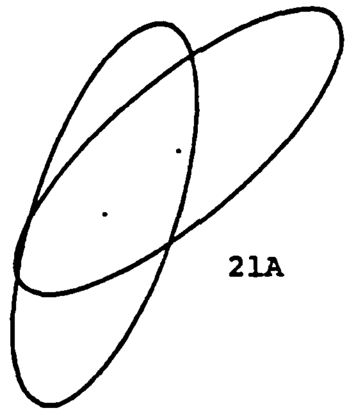
The shape and location of A, B, and C can be chosen so that for order of testing:

Test Results	Order of Testing/Combination	
	A, B, C	A, C, B
1st Test	$T(A,B) = 0$	$T(A,C) = k$
2nd Test	$T(A \oplus B, C) = 2k$	$T(A \oplus C, B) = k$

These results state that A and B have exactly the same location; and that when testing is done in the order A, B, C the ellipses $A \oplus B$ and C will not be accepted for combination. When testing is done in the order A, C, B, however, all three ellipses are combined.

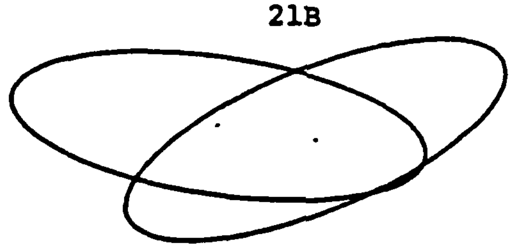
There is no mathematical preference for either order in the cases discussed above. That is, there is no reason, just from the math, why one should prefer to test in the order A, B, C over A, C, B, and vice versa. The only basis for making this decision is knowledge of which ellipses actually should be combined--which would make the test unnecessary. However, it is unclear how often cases like this occur. While the errors themselves are significant, the chance of making them may not be.

FIGURE 21



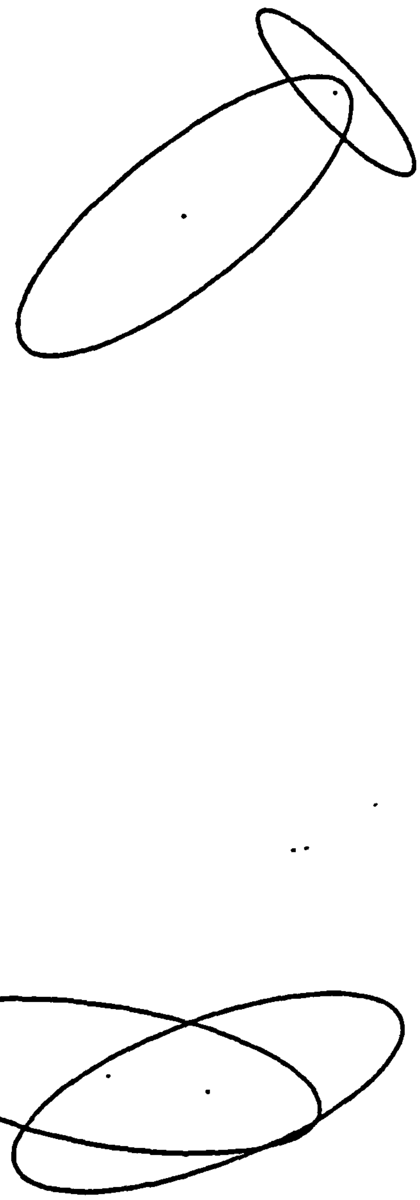
21A

21B



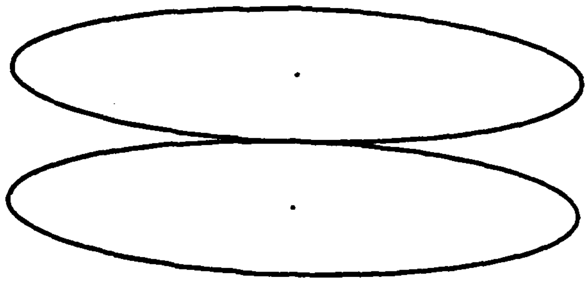
Rotation does not affect the test.

FIGURE 22

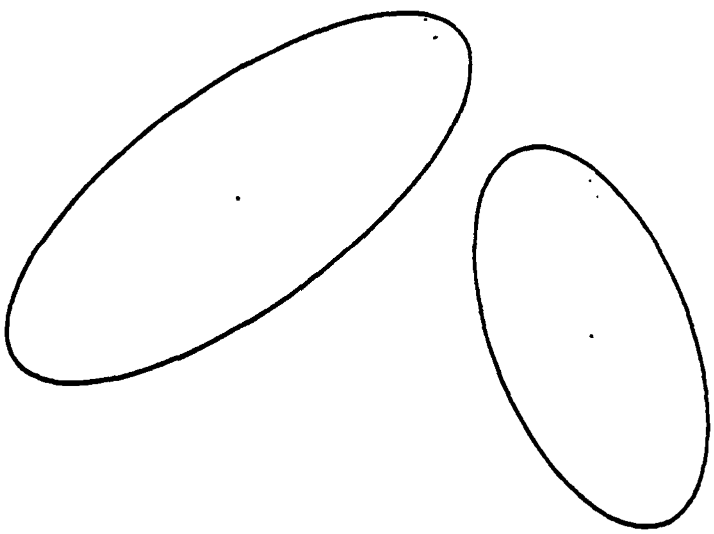


If one of the ellipses contains the center point of the other, the test will accept.

FIGURE 23

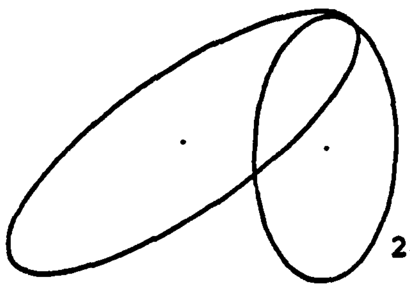


If the ellipses are tangent, the test will reject.



If the ellipses do not intersect, the test will reject.

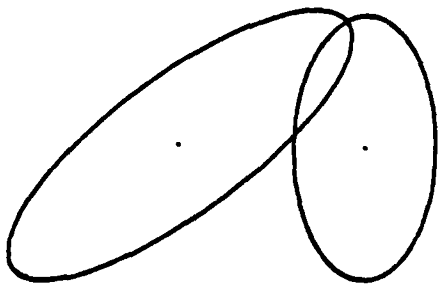
FIGURE 24



24A

Neither of these ellipses contains the other's center, but the test accepts.

24B



These ellipses intersect, but the test rejects.

V. SIMULATION RESULTS

In the previous sections we have discussed the "goodness" of the combination method and statistical test in mathematical terms. For example, we have shown that the results of the combination method are not affected by the order of combination; any number of ellipses may be combined in any order and the same resultant ellipse will be gained. However, we have also shown that order of combination can have a serious effect on the results of the statistical test.

In this section one aspect of the "goodness" of these methods will be examined. More specifically, we wish to know if the resultant ellipse sometimes misses the true location of the emitter, and if so how often. We will use simulation to study this question. Simulation involves running an experiment, such as finding out if the resultant ellipse contains the true location, a large number of times, so that the chance of it happening may be estimated. Thus, a computer will simulate data coming in from sensors in the field, and generate many confidence ellipse and point estimates. The combination method and statistical test will be applied to these estimates, and the results compiled.

Before the results of the simulations can be discussed, however, the possible errors and the simulation procedures must be described.

The simulations use a simple model, containing either one emitter or two emitters positioned close together, and two sets of sensors, A and B. Each emitter is some radio source such as a radar unit. By set of sensors, we mean an array of sensors working together to get a fix on an emitter. The computer generates location data from each set of sensors. If there are two emitters, then sensor set A is sighting one emitter and sensor set B is sighting the other. If there is only one emitter, then both are sighting it.

Each set of sensors gathers data about the location of the emitter it is sighting, and sends the data into the analysis center in the form of a confidence ellipse and point estimate. In the analysis which follows it is the ellipse and point estimates which are simulated, as opposed to the actual sensor readings. The estimates are compared using the statistical test, and if the test accepts that there is only one emitter the ellipses and point estimates may be combined.

There are a number of errors which may occur during this process. First of all, the statistical test may decide incorrectly. That is, it may decide that there are two emitters when there is only one (Type I error), or it may decide that there is only one emitter when there are in fact two (Type II error). As the labels Type I and Type II error imply, these errors are common, and may occur whenever a statistical test is used.

Another sort of error involves the combined estimate. Suppose that there is only one emitter, and the statistical test accepts, deciding correctly. Then the combination method may correctly be used to combine the two ellipse and point estimates. There is a possibility that the combined ellipse will not contain the true location of the emitter. Since the ellipse is used to show the area in which the emitter is likely to be found, this error is quite serious.

A more complex situation results if there are two emitters but the test

decides that there is only one emitter. In this case one would hope that the combined ellipse would contain both emitters, in order to avoid compounding the error.

Thus, we are interested in the following cases:

A. How often the statistical test accepts that there is one emitter.
B. If it accepts, how often the combined ellipse contains the emitter(s).

It is simple to determine mathematically how often the statistical test should accept that there is one emitter, but how often the latter case occurs is more difficult. Simulation is useful here. By using the computer to generate the data, the statistical test and combination method may be applied many times. Each time one of the errors discussed above is made, it is recorded. Thus, the probability of making these errors can be estimated.

The first step is choosing parameters for the data gathered by sensor sets A and B. The program assumes that there are two emitters, and that the data gathered by each set follows a bivariate normal distribution about the true location of the emitter being sighted by the sensor set. The variance-covariance matrices of the data are specified, as are the locations of the emitters. The case in which there is a single emitter is specified by making the two emitter locations equal. Also specified are the number of observations each set of sensors will use to calculate its confidence ellipses.

In the second step, data sets of the specified sample size are generated, and the statistical test is performed. If the test accepts that there is only one emitter, then the ellipses are combined, and it is determined whether the resultant ellipse contains either or both of the true emitter locations. In the results discussed later in this section, this step was repeated 500 times.

Finally, the results are totalled and compared to the total number of simulations (here, 500). In this way probabilities may be estimated. For example, if the test accepts 475 times out of 500, there is about a 95% chance that the test will accept in this case.

The compiled results are listed in Tables 1-4. Consider Table 1. The parameters for the runs contained in Table 1 are listed in the heading. Sensor set A generated 5 observations about the true location of Emitter 1 at (0,0), with covariance matrix S_A for each simulation. Also, Sensor set B generated 5 observations about the true location of Emitter 2 with covariance matrix S_B.

The Results

The first column of the table show the location of Emitter 2. When Emitter 1 and Emitter 2 are at the same location (0,0), this is equivalent to there being only one emitter.

The second column shows how often the statistical test accepted that there was only one emitter. In Table 1 this ranges from 97.4% of the time to 54.4% of the time, which are reasonable values given the parameters listed at the top of the table.

Columns three and four show how often the combined ellipse does not contain the true location of Emitter 1, Emitter 2, and both. Column three expresses this as a percentage of the total simulations, and column four as a

percentage of the number of times the statistical test accepted. For example, when Emitter 1 and Emitter 2 are the same, in 4.6% of the 500 simulations the combined ellipse did not contain the true location of the emitter. This is 4.7% of the total number of times the statistical test accepted that there was only one emitter. (Remember that ellipses are only combined if the test accepts).

Column five is a key showing what is in each block of percentages. For example, when the location of Emitter 2 is (.5, .5), Emitter 1 is not in the combined ellipse 4% of the time, Emitter 2 is not in the ellipse 4% of the time, and neither one is in the ellipse 2% of the time. Note that there is double counting here; for example, the 2% of the time that neither one is in the ellipse is also counted in the 4% of the time that Emitter 1 is not in the ellipse.

Tables 2, 3 and 4 are formatted identically.

Discussion of the Results

The statistical test used in the simulations was a 95% test. This means that when there really is only one emitter, the chance of saying that there are two is limited to 5% (100%-95%). As stated above, this is called Type I error. Indeed, Tables 1-4 indicate that when there is only one emitter, the test accepts approximately 95% of the time. This is usual and reasonable. However, there is an additional error: sometimes when the test accepts, the combined ellipse does not contain the true emitter location. The tables indicate that this occurs approximately 5% of the time as well. Since this error can only happen when the test has accepted, this means that the results are in error about 10% of the time--5% for this and 5% incorrect rejections.

This may be a significant problem.

It is also enlightening to examine what happens when there are really two emitters. Since there are two emitters, it is an error to accept there is only one (Type II error). It is better to reject, and deal with each emitter separately. However, the test will sometimes accept incorrectly, and combine the confidence ellipses. From a practical standpoint, it is desirable that the combined ellipse contains both emitter locations in this case. If one of the locations falls outside the combined ellipse, the analysis center essentially has no idea of its whereabouts. If both locations are in the ellipse, at least both are in a region where the analysis center expects to find an emitter.

Examine the case in Table 2 where Emitter 2 is located at (1.5,1.5). Here there is approximately a 55% chance of committing Type II error, since that is how often the test accepted. Note that in almost every case in which the test accepted, one or the other emitter was not contained in the combined ellipse (88% of the time). About half of the combined ellipses contained Emitter 1, and about half contained Emitter 2.

Similar results to these may be found in each of the tables.

It is uncertain how often this problem will arise, since it is dependent upon two emitters being located very close together. "Close" is with respect to the possible spread in the fixes on each emitter's location. It may be that signal parametrics might help in distinguishing between them as well.

TABLE 1:
ELLIPSE TEST AND COMBINATION SIMULATION RESULTS

Simulations = 500
Sample sizes = (5, 5)
Using TRUE covariance matrices
Location of Emitter 1 = (0,0)

$S_A = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$, $S_B = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$

Location of Emitter 2	Accepts % of Total	Accepts not in % of Total	Accepts not in % of Accepts	Emitter Location not in Combined Ellipse
(0, 0)	97.4%	4.6%	4.7%	1 Not In
		4.6%	4.7%	2 Not In
		4.6%	4.7%	Both Not In
(.5, .5)	93.0%	4.0%	4.3%	1 Not In
		4.0%	4.3%	2 Not In
		2.0%	2.2%	Both Not In
(1, 1)	92.0%	7.6%	8.3%	1 Not In
		7.6%	8.3%	2 Not In
		1.8%	1.9%	Both Not In
(1.5, 1.5)	84.8%	11.8%	13.9%	1 Not In
		13.6%	16.0%	2 Not In
		1.8%	2.1%	Both Not In
(2, 2)	75.2%	14.6%	19.4%	1 Not In
		19.2%	25.5%	2 Not In
		0.8%	1.1%	Both Not In
(2.5, 2.5)	68.0%	20.2%	29.7%	1 Not In
		25.0%	36.8%	2 Not In
		3.2%	3.2%	Both Not In
(3, 3)	54.4%	22.4%	41.2%	1 Not In
		27.2%	50.0%	2 Not In
		3.8%	7.0%	Both Not In

TABLE 2:
ELLIPSE TEST AND COMBINATION SIMULATION RESULTS

Simulations = 500
Sample sizes = (20, 20)
Using TRUE covariance matrices
Location of Emitter 1 = (0,0)

$S_A = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$, $S_B = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$

Location of Emitter 2	Accepts % of Total	Accepts not in % of Total	Accepts not in % of Accepts	Emitter Location not in Combined Ellipse
		5.0%	5.3%	1 Not In
(0, 0)	94.4%	5.0%	5.3%	2 Not In
		5.0%	5.3%	Both Not In
		10.8%	12.0%	1 Not In
(.5, .5)	89.9%	6.8%	7.6%	2 Not In
		1.6%	1.8%	Both Not In
		18.0%	23.5%	1 Not In
(1, 1)	76.6%	17.0%	22.2%	2 Not In
		2.0%	2.6%	Both Not In
		25.2%	46.0%	1 Not In
(1.5, 1.5)	54.8%	26.4%	48.2%	2 Not In
		3.4%	6.2%	Both Not In
		22.4%	72.7%	1 Not In
(2, 2)	30.8%	22.2%	72.1%	2 Not In
		13.8%	44.8%	Both Not In
		9.4%	95.9%	1 Not In
(2.5, 2.5)	9.8%	8.6%	87.8%	2 Not In
		8.2%	83.7%	Both Not In
		4.0%	100.0%	1 Not In
(3, 3)	4.0%	3.8%	95.0%	2 Not In
		3.8%	95.0%	Both Not In

TABLE 3:
ELLIPSE TEST AND COMBINATION SIMULATION RESULTS

Simulations = 500
Sample sizes = (5, 5)
Using TRUE covariance matrices
Location of Emitter 1 = (0,0)

$S_A = \begin{pmatrix} 10 & 0 \\ 0 & 2 \end{pmatrix}, \quad S_B = \begin{pmatrix} 2 & 0 \\ 0 & 10 \end{pmatrix}$

Location of Emitter 2	Accepts % of Total	Accepts not in % of Total	Accepts not in % of Accepts	Emitter Location not in Combined Ellipse
(0, 0)	96.0%	4.6%	4.8%	1 Not In
		4.6%	4.8%	2 Not In
		4.6%	4.8%	Both Not In
(.5, .5)	93.4%	9.4%	10.1%	1 Not In
		7.8%	8.4%	2 Not In
		2.4%	2.6%	Both Not In
(1, 1)	89.8%	19.6%	21.8%	1 Not In
		21.8%	24.3%	2 Not In
		5.4%	6.0%	Both Not In
(1.5, 1.5)	78.2%	36.4%	46.5%	1 Not In
		40.0%	51.2%	2 Not In
		13.2%	16.9%	Both Not In
(2, 2)	63.0%	47.8%	75.9%	1 Not In
		50.0%	79.4%	2 Not In
		34.8%	34.8%	Both Not In
(2.5, 2.5)	46.4%	43.2%	93.1%	1 Not In
		42.4%	91.4%	2 Not In
		39.2%	84.5%	Both Not In
(3, 3)	29.8%	28.6%	95.9%	1 Not In
		29.6%	99.3%	2 Not In
		28.4%	95.3%	Both Not In

TABLE 4:
ELLIPSE TEST AND COMBINATION SIMULATION RESULTS

Simulations = 500
Sample sizes = (20, 20)
Using TRUE covariance matrices
Location of Emitter 1 = (0,0)

$S_A = \begin{pmatrix} 10 & 0 \\ 0 & 2 \end{pmatrix}$, $S_B = \begin{pmatrix} 2 & 0 \\ 0 & 10 \end{pmatrix}$

Location of Emitter 2	Accepts % of Total	Accepts not in % of Total	Accepts not in % of Accepts	Emitter Location not in Combined Ellipse
		5.6%	5.9%	1 Not In
(0, 0)	94.4%	5.6%	5.9%	2 Not In
		5.6%	5.9%	Both Not In
		23.4%	25.8%	1 Not In
(.5, .5)	90.6%	20.8%	23.0%	2 Not In
		5.8%	6.4%	Both Not In
		45.8%	76.1%	1 Not In
(1, 1)	60.2%	44.8%	74.4%	2 Not In
		30.4%	50.5%	Both Not In
		31.8%	98.8%	1 Not In
(1.5, 1.5)	32.2%	31.8%	98.8%	2 Not In
		30.8%	95.7%	Both Not In
		9.6%	100.0%	1 Not In
(2, 2)	9.6%	9.6%	100.0%	2 Not In
		9.6%	100.0%	Both Not In
		1.2%	100.0%	1 Not In
(2.5, 2.5)	1.2%	1.2%	100.0%	2 Not In
		1.2%	100.0%	Both Not In
		0.0%	—	1 Not In
(3, 3)	0.0%	0.0%	—	2 Not In
		0.0%	—	Both Not In

MATHEMATICAL APPENDIX

DEFINITIONS:

X denotes the coordinates of an arbitrary point.

X_i denotes the coordinates of the center of the ith ellipse (i.e. the ith estimate of target location.)

S_i denotes the covariance matrix of the ith ellipse.

E_i denotes the set of points in the ith ellipse $=\{X|(X-X_i)^T S_i^{-1}(X-X_i)\le k\}$ where k denotes the confidence cut-off value being used.

D_i denotes an ellipse similar to E_i. $D_i=\{X|(X-X_i)^T S_i^{-1}(X-X_i)\le k/2\}$

$\oplus$ denotes an operator on the X_i, S_i, and E_i defined so that

$$S_i \oplus S_j=(S_i^{-1}+S_j^{-1})^{-1}$$
$$X_i \oplus X_j=(S_i \oplus S_j)(S_i^{-1}X_i+S_j^{-1}X_j)$$
$$E_i \oplus E_j=\{X|(X-(X_i \oplus X_j))^T(S_i \oplus S_j)^{-1}(X-(X_i \oplus X_j))<k\}$$

Note that $\oplus$ is a mathematical representation of a method of combining ellipses.

$T(E_i,E_j)$ denotes the value of the acceptance test statistic.

$$T(E_i,E_j) = (X_i-X_j)^T(S_i+S_j)^{-1}(X_i-X_j)$$

PROPERTY 1: $\oplus$ is commutative.

Proof. $S_i \oplus S_j=(S_i^{-1}+S_j^{-1})^{-1}=(S_j^{-1}+S_i^{-1})^{-1}=S_j \oplus S_i$.
$X_i \oplus X_j=(S_i \oplus S_j)(S_i^{-1}X_i+S_j^{-1}X_j)=(S_j \oplus S_i)(S_j^{-1}X_j+S_i^{-1}X_i)=X_j \oplus X_i$.

PROPERTY 2: $\oplus$ is associative. (Together with Property 1 this implies that order of computation does not matter.) $\oplus$ generalizes to combining n ellipses in the following natural way.

$$\bigoplus_{i=1}^{n} S_i=\left(\sum_{i=1}^{n} S_i^{-1}\right)^{-1}$$
$$\bigoplus_{i=1}^{n} X_i=\left(\bigoplus_{i=1}^{n} S_i\right)\left(\sum_{i=1}^{n} S_i^{-1}X_i\right)$$

Proof. Associativity follows directly from these generalized definitions, which will be validated by induction. When $n=2$, the preceding definitions are the original definitions for $\oplus$. Now, assume they are valid for $n=k$. It must now be shown that they are valid when we combine the k+1st ellipse with the previous combination of k ellipses.

$$\bigoplus_{i=1}^{k+1} S_i=\left(\bigoplus_{i=1}^{k} S_i\right)\oplus S_{k+1}=\left(\sum_{i=1}^{k} S_i^{-1}\right)^{-1}\oplus S_{k+1}=\left(\left(\sum_{i=1}^{k} S_i^{-1}\right)+S_{k+1}^{-1}\right)^{-1}=\left(\sum_{i=1}^{k+1} S_i^{-1}\right)^{-1}.$$

Thus it is true for covariance matrices. For target location we have

$$\overset{k+1}{\underset{i=1}{\oplus}} X_i = (\overset{k}{\underset{i=1}{\oplus}} X_i) \oplus X_{k+1}$$
$$= ((\overset{k}{\underset{i=1}{\oplus}} S_i) \oplus S_{k+1})((\overset{k}{\underset{i=1}{\oplus}} S_i)^{-1}(\overset{k}{\underset{i=1}{\oplus}} X_i) + S_{k+1}^{-1} X_{k+1})$$
$$= (\overset{k+1}{\underset{i=1}{\oplus}} S_i)((\overset{k}{\underset{i=1}{\oplus}} S_i)^{-1}(\overset{k}{\underset{i=1}{\oplus}} S_i)(\overset{k}{\underset{i=1}{\Sigma}} S_i^{-1} X_i) + S_{k+1}^{-1} X_{k+1})$$
$$= (\overset{k+1}{\underset{i=1}{\oplus}} S_i)(\overset{k+1}{\underset{i=1}{\Sigma}} S_i^{-1} X_i).$$

<u>PROPERTY 3:</u> The shape of the resultant ellipse depends only on the shape of the original ellipses (Note that the shape of any ellipse E depends only on the shape of its covariance matrix S).

Proof. The formula for $S_i \oplus S_j$ depends only on S_i and S_j and not on X_i or X_j.

<u>PROPERTY 4:</u> If $X_i = X_j$ then $X_i \oplus X_j = X_i$.

Proof.
$$\begin{aligned} X_i \oplus X_j &= (S_i \oplus S_j)(S_i^{-1} X_i + S_j^{-1} X_j) \\ &= (S_i^{-1} + S_j^{-1})^{-1}(S_i^{-1} X_i + S_j^{-1} X_i) \\ &= (S_i^{-1} + S_j^{-1})^{-1}(S_i^{-1} + S_j^{-1}) X_i \\ &= X_i . \end{aligned}$$

<u>PROPERTY 5:</u> If $X_i = X_j$ then $E_i \oplus E_j$ lies entirely inside the intersection of E_i and E_j.

Proof.
$$\begin{aligned} X \in E_i \oplus E_j &\Rightarrow (X - X_i \oplus X_j)^T (S_i \oplus S_j)^{-1} (X - X_i \oplus X_j) \le k \\ &\Rightarrow (X - X_i)^T (S_i^{-1} + S_j^{-1})(X - X_i) \le k \\ &\Rightarrow (X - X_i)^T S_i^{-1} (X - X_i) \le k \\ &\Rightarrow X \in E_i \end{aligned}$$

Thus $E_i \oplus E_j$ is contained in E_i. Containment in E_j is similar.

<u>PROPERTY 6:</u> If $X_i = X_j$ then $E_i \oplus E_j$ contains the intersection of D_i and D_j. Note that the D ellipses are proportionally smaller (.707) versions of the E ellipses.

Proof.
$$\begin{aligned} X \in D_i &\Rightarrow (X - X_i)^T S_i^{-1} (X - X_i) \le k/2 \\ &\Rightarrow (X - X_i \oplus X_j)^T S_i^{-1} (X - X_i \oplus X_j) \le k/2 \end{aligned}$$

Similarly $X \in D_j \Rightarrow (X - X_i \oplus X_j)^T S_j^{-1} (X - X_i \oplus X_j) \le k/2$

If X is in the intersection of D_i and D_j then $X \in D_i$ and $X \in D_j$. Since both inequalities hold in this case they may be added to get that X satisfies

$$(X - X_i \oplus X_j)^T (S_i^{-1} + S_j^{-1})(X - X_i \oplus X_j) \le k/2 + k/2$$
$$\Rightarrow \quad (X - X_i \oplus X_j)^T (S_i \oplus S_j)^{-1} (X - X_i \oplus X_j) \le k$$

$\Rightarrow X \in E_i \oplus E_j$.

<u>PROPERTY 7:</u> $X_i \oplus X_j$ is a point of tangency between $(X-X_i)^T S_i^{-1}(X-X_i)=\alpha_i$ and $(X-X_j)^T S_j^{-1}(X-X_j)=\alpha_j$ where

$$\alpha_i=(X_i \oplus X_j - X_i)^T S_i^{-1}(X_i \oplus X_j - X_i)$$
$$\alpha_j=(X_i \oplus X_j - X_j)^T S_j^{-1}(X_i \oplus X_j - X_j).$$

Proof. The gradients of the two curves respectively are

$$\mathrm{GRAD}_i=2S_i^{-1}(X-X_i) \qquad \mathrm{GRAD}_j=2S_j^{-1}(X-X_j)$$

At $X_i \oplus X_j$, $\mathrm{GRAD}_i=2S_i^{-1}((S_i \oplus S_j)(S_i^{-1}X_i+S_j^{-1}X_j)-X_i)$

$$=2S_i^{-1}((S_i^{-1}+S_j^{-1})^{-1}(S_i^{-1}X_i+S_j^{-1}X_j)-X_i)$$
$$=2S_i^{-1}\ (S_i^{-1}+S_j^{-1})^{-1}((S_i^{-1}X_i+S_j^{-1}X_j)-(S_i^{-1}+S_j^{-1})X_i)$$

$$=2S_i^{-1}(S_i^{-1}+S_j^{-1})^{-1}S_j^{-1}(X_j-X_i)$$
$$=2(S_iS_i^{-1}S_j+S_iS_j^{-1}S_j)^{-1}(X_j-X_i)$$
$$=2(S_j+S_i)^{-1}(X_j-X_i)$$

Similarly at $X_i \oplus X_j$ $\mathrm{GRAD}_j=2(S_i+S_j)^{-1}(X_i-X_j)=-\mathrm{GRAD}_i$. Hence tangency.

<u>PROPERTY 8:</u> $X_i \oplus X_j$ is the point where $(X-X_i)^T S_i^{-1}(X-X_i)+(X-X_j)^T S_j^{-1}(X-X_j)$ is a minimum.

Proof. The derivative of this expression using the notation of the previous proof is $\mathrm{GRAD}_i+\mathrm{GRAD}_j$. The previous proof shows this sum is 0.

Note that since the joint likelihood is a function of the quadratic form minimized in PROPERTY 8, it is implied that $\oplus$ yields a maximum likelihood estimate of the associated normal.

<u>PROPERTY 9:</u> Given α such that $(X-X_i)^T S_i^{-1}(X-X_i)=\alpha$ and $(X-X_j)^T S_j^{-1}(X-X_j)=\alpha$ are tangent, then $X=X_i \oplus X_j$ satisfies $(X-X_i)^T S_i^{-1}(X-X_i)\le 2\alpha$ and $(X-X_j)^T S_j^{-1}(X-X_j)\le 2\alpha$.

Proof. By PROPERTY 8 the value of $(X-X_i)^T S_i^{-1}(X-X_i)+(X-X_j)^T S_j^{-1}(X-X_j)$ can not be smaller at the point of tangency than it is at $X_i \oplus X_j$ where the value is $\alpha+\alpha$ or 2α.

<u>DEFINITION:</u> For any change of coordinates of the form

$$\phi(X)=AX+Y$$

where A is any invertible matrix and Y is a constant vector, define

$$\phi(S)=ASA^T \quad \text{for covariance matrices S.}$$

<u>PROPERTY 10:</u> Change of coordinates have the following properties:

(1) $\phi^{-1}(X)=A^{-1}X-A^{-1}Y$ and $\phi^{-1}(S)=A^{-1}S(A^{-1})^T$.

(2) E_i is 'preserved' under this change of coordinates, i.e.
$\{X|(X-X_i)^T S_i^{-1}(X-X_i)\leq k\}=\{X|(\phi(X)-\phi(X_i))^T\phi(S_i)^{-1}(\phi(X)-\phi(X_i))\leq k\}$

(3) $\phi(S_i \oplus S_j)=\phi(S_i)\oplus\phi(S_j)$.

(4) $\phi(X_i \oplus X_j)=\phi(X_i)\oplus\phi(X_j)$.

(5) The test to accept/reject combination of E_i and E_j is independent of the coordinate system it is calculated in.

Proof. (1) It suffices to show that $\phi(\phi^{-1})=I$.

For X note that $\phi(\phi^{-1}(X))=\phi(A^{-1}X-A^{-1}Y)=A(A^{-1}X-A^{-1}Y)+Y=X-Y+Y=X$.

For S note that $\phi(\phi^{-1}(S))=\phi(A^{-1}S(A^{-1})^T)=A(A^{-1}S(A^{-1})^T)A^T=S$.

$$
\begin{aligned}
(2)\ &(\phi(X)-\phi(X_i))^T\phi(S_i)^{-1}(\phi(X)-\phi(X_i))\\
&=((AX+Y)-(AX_i+Y))^T(AS_iA^T)^{-1}((AX+Y)-(AX_i+Y))\\
&=(A(X-X_i))^T(A^T)^{-1}S_i^{-1}A^{-1}(A(X-X_i))\\
&=(X-X_i)^TA^T(A^T)^{-1}S_i^{-1}A^{-1}A(X-X_i)\\
&=(X-X_i)^TS_i^{-1}(X-X_i).
\end{aligned}
$$

$$
\begin{aligned}
(3)\ \phi(S_i\oplus S_j)&=A(S_i^{-1}+S_j^{-1})^{-1}A^T\\
&=((A^T)^{-1}(S_i^{-1}+S_j^{-1})^{-1}A^{-1})^{-1}\\
&=((A^T)^{-1}S_i^{-1}A^{-1}+(A^T)^{-1}S_j^{-1}A^{-1})^{-1}\\
&=((AS_iA^T)^{-1}+(AS_jA^T)^{-1})^{-1}\\
&=(\phi(S_i)^{-1}+\phi(S_j)^{-1})^{-1}\\
&=\phi(S_i)\oplus\phi(S_j).
\end{aligned}
$$

$$
\begin{aligned}
(4)\ \phi(X_i\oplus X_j)&=A(X_i\oplus X_j)+Y\\
&=A(S_i\oplus S_j)(S_i^{-1}X_i+S_j^{-1}X_j)+Y\\
&=A(S_i\oplus S_j)(S_i^{-1}X_i+S_j^{-1}X_j)+A(S_i\oplus S_j)(S_i\oplus S_j)^{-1}A^{-1}Y\\
&=A(S_i\oplus S_j)A^T(A^T)^{-1}(S_i^{-1}X_i+S_j^{-1}X_j+(S_i^{-1}+S_j^{-1})A^{-1}Y)\\
&=\phi(S_i\oplus S_j)(A^T)^{-1}(S_i^{-1}A^{-1}(AX_i+Y)+S_j^{-1}A^{-1}(AX_j+Y))\\
&=\phi(S_i)\oplus\phi(S_j)((AS_iA^T)^{-1}\phi(X_i)+(AS_jA^T)^{-1}\phi(X_j))\\
&=\phi(S_i)\oplus\phi(S_j)(\phi(S_i)^{-1}\phi(X_i)+\phi(S_j)^{-1}\phi(X_j))\\
&=\phi(X_i)\oplus\phi(X_j).
\end{aligned}
$$

$$
\begin{aligned}
(5)\ &(\phi(X_i)-\phi(X_j))^T(\phi(S_i)+\phi(S_j))^{-1}(\phi(X_i)-\phi(X_j))\\
&=((AX_i+Y)-(AX_j+Y))^T(AS_iA^T+AS_jA^T)^{-1}((AX_i+Y)-(AX_j+Y))\\
&=(X_i-X_j)^TA^T(A^T)^{-1}(S_i+S_j)^{-1}A^{-1}A(X_i-X_j)\\
&=(X_i-X_j)^T(S_i+S_j)^{-1}(X_i-X_j).
\end{aligned}
$$

The fact that properties of interest are preserved under transformations of the type specified by ϕ above shall be used to simplify the analysis in the proofs that follow. For more details on the relationships between these transformations and symmetric quadratic forms (such as ellipses) and the underlying geometry see Metric Affine Geometry by Snapper and Troyer. Some

readers may have encountered this material in connection with the study of the mathematics of Minkowski space and it is the pages immediately preceeding this example that are of the most interest here. In general a quadratic form may be transformed into a diagonal matrix of 1's, 0's and -1's. Since the quadratic forms of interest are ellipses only 1's are possible.

For the purposes of proofs one of the center points may be sent to the origin and one of the covariance matrices made into the identity matrix by use of these transformations. The one qualification is that the results that are demonstrated must be transformed back into the original space if shape is used in the result.

PROPERTY 11: If the center of one ellipse is in the second ellipse then the acceptance test will accept. On the other hand, if the ellipses do not intersect at more than one point then the test will reject.

Proof. For the first part of the proof the space will be transformed so that

$X_1=(0,0)$ $X_2=(a,0)$ $S_1=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ $S_2=\begin{pmatrix}b&c\\c&d\end{pmatrix}$.

Note that E_1 has been transformed into a circle.

First it must be shown that $X_1 \varepsilon E_2 \Rightarrow (X_2-X_1)^T(S_1+S_2)^{-1}(X_1-X_2) \le k$

$$\begin{aligned}(X_1-X_2)^T(S_1+S_2)^{-1}(X_1-X_2) &= a^2(d+1)/((b+1)(d+1)-c^2)\\ &\le a^2d/(bd-c^2)\\ &= (X_1-X_2)^T S_2^{-1}(X_1-X_2)\\ &\le k \qquad \text{(since } X_1 \ \varepsilon\ E_2.)\end{aligned}$$

Consequently, the hypothesis test will accept.

For the second part of the proof, the space will be transformed as follows, without loss of generality.

$X_1=\begin{pmatrix}0\\0\end{pmatrix}$, $S_1=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $S_2^{-1}=\begin{pmatrix}a&b\\b&c\end{pmatrix}$.

Further, the two ellipses are assumed to be tangent (intersecting only at one point), where the point of tangency P is as follows:

$P=\begin{pmatrix}\sqrt{k}\\0\end{pmatrix}$, where k is the cutoff value defined above.

The normal vector to an ellipse at X is of the form $S_i^{-1}(X_i-X)$.

The normal vector to E_1 at point P is $S_1^{-1}(X_1-P) = I(0-P) = -P$, which is proportional to the vector $\begin{pmatrix}1\\0\end{pmatrix}$. Since E_1 and E_2 are tangent at P, the normal to E_2 is also proportional to $\begin{pmatrix}1\\0\end{pmatrix}$, and is thus perpendicular to the vector $\begin{pmatrix}0\\1\end{pmatrix}$. Since the normal to E_2 at P is $S_2^{-1}(X_2-P)$, we know that $(0,\ 1)S_2^{-1}(X_2-P)=0$.

Thus $(b,\ c)(X_2-P)=0$, and X_2-P is proportional to the vector $\begin{pmatrix}-c\\b\end{pmatrix}$. This implies that $X_2=P+\begin{pmatrix}-c\\b\end{pmatrix}t$ for some t.

We find the value of t as follows, letting $D = \text{Det } S_2$:

$$(X_2-P)^T S_2^{-1}(X_2-P)=k$$

$$\Rightarrow \quad t^2\begin{pmatrix}-c\\ b\end{pmatrix}^T\begin{pmatrix}a & b\\ b & c\end{pmatrix}\begin{pmatrix}-c\\ b\end{pmatrix}=k$$

$$\Rightarrow \quad t^2(-D,\ 0)\begin{pmatrix}-c\\ b\end{pmatrix}=k$$

Solving for t results in $t=\pm\sqrt{k/(cD)}$. The negative root is the one with the correct geometry for exterior tangency.

The test will only accept if $(X_1-X_2)^T(I+S_2)^{-1}(X_1-X_2)\le k$. Substituting in known information results in $(-X_2)^T(I+S_2)^{-1}(-X_2)\le k$.

Now $-\sqrt{k}$ may be factored out of X_2 both times, allowing the k on the right to be cancelled. Thus the test will accept if

$$\left(\begin{pmatrix}1\\ 0\end{pmatrix}-\begin{pmatrix}-c\\ b\end{pmatrix}(1/\sqrt{cD})\right)^T\left(I+\begin{pmatrix}c & -b\\ -b & a\end{pmatrix}(1/D)\right)^{-1}\left(\begin{pmatrix}1\\ 0\end{pmatrix}-\begin{pmatrix}-c\\ b\end{pmatrix}(1/\sqrt{cD})\right)\le 1$$

Simplifying the above inequality, we get

$$\left(\begin{pmatrix}\sqrt{D}\\ 0\end{pmatrix}-\begin{pmatrix}-c\\ b\end{pmatrix}(1/\sqrt{c})\right)^T\begin{pmatrix}c+D & -b\\ -b & a+D\end{pmatrix}^{-1}\left(\begin{pmatrix}\sqrt{D}\\ 0\end{pmatrix}-\begin{pmatrix}-c\\ b\end{pmatrix}(1/\sqrt{c})\right)\le 1$$

$$\Leftrightarrow \quad \left(\begin{pmatrix}\sqrt{D}\\ 0\end{pmatrix}-\begin{pmatrix}-c\\ b\end{pmatrix}(1/\sqrt{c})\right)^T\begin{pmatrix}a+D & b\\ b & c+d\end{pmatrix}\left(\begin{pmatrix}\sqrt{D}\\ 0\end{pmatrix}-\begin{pmatrix}-c\\ b\end{pmatrix}(1/\sqrt{c})\right)\le(a+D)(c+D)-b^2$$

$$\Leftrightarrow \quad \left(\begin{pmatrix}\sqrt{D}\\ 0\end{pmatrix}-\begin{pmatrix}-c\\ b\end{pmatrix}(1/\sqrt{c})\right)^T\left(\sqrt{D}\begin{pmatrix}a+D\\ b\end{pmatrix}-(1/\sqrt{c})\begin{pmatrix}b^2-ac-cD\\ bD\end{pmatrix}\right)\le(a+D)(c+D)-b^2$$

$$\Leftrightarrow \quad D(a+D)-\sqrt{D/c}(-D-cD)-\sqrt{D/c}(-ca-cD+b^2)+(1/c)(-c(-D-cD)+b^2D)\le(a+D)(c+D)-b^2$$

(In making the last simplification, remember that $D = \text{Det } S_2 = ac-b^2$).

$$\Leftrightarrow \quad D(a+D)+2\sqrt{D/c}(D+cD)+(1/c)[c(c+1)+b^2]D\le ac+(a+c)D+D^2-b^2$$

$$\Leftrightarrow \quad aD+2\sqrt{D/c}(c+1)D+(1/c)(c^2+b^2)D\le(a+c)D$$

$$\Leftrightarrow \quad a+2\sqrt{D/c}(c+1)+(1/c)(c^2+b^2)\le a+c$$

Finally, the test will accept if the ellipses have an exterior tangency and

$$(b^2/c)+2\sqrt{D/c}(c+1)\le 0.$$

Since c and D are always greater than 0, this inequality is never satisfied. Thus, if the ellipses have an exterior tangency, the test will reject. From this it follows that if the ellipses have no intersection, the test will also reject.

The next result is not actually geometrical but is included here because of its general significance and because the tools needed to prove it are available in this appendix.

PROPERTY 12: Given a sequence of ellipses to be combined $E_1, E_2, \ldots, E_n$ if none were rejected then the sum of accept-reject test conditions computed at each step

$$\sum_{k=2}^{n} T\left(\bigoplus_{i=1}^{k-1} E_i, E_k\right)$$

comes out the same regardless of the order of the E_i. (With rejections the result does not have the form shown above because of missing terms and is not independent of order.)

Proof. Proceed by by induction. For n=2 the symmetry of the individual test gives the result. If the result is assumed to be true for $n \leq m$ then the n=m+1 case must be shown. Note that for this case any permutation that does not move the m+1st ellipse would have the same k=m+1st term in the sum so that it is sufficient to compare the value of the sum for k=2 to m. But since that sum applies to the same k ellipses our induction hypothesis applies. Thus in the case where the m+1st ellipse is not moved our induction criteria is satisfied.

The fact that the induction result holds when the position of the last ellipse is not permuted will allow restriction of our study to a single case. The reason for this simplification follows. If the result holds for two permutations α and β then it holds for their composition. Thus it is sufficient to prove that the generating set consisting of exchanges of adjacent ellipses each preserve the sum. Only one of these permutations moves the last ellipse. Thus it is sufficient to show that exchanging the last two ellipses preserves the sum.

Then, if there are m ellipses, the first m-2 are unchanged when the last two are exchanged. As these m-2 ellipses are constant, they may be combined into a single ellipse. The resulting expression is the k=3 case with the new first ellipse being the same as the old combination of the first m-2 ellipses. Now if we use the linear transformation sending X_1 to the origin and S_1 to the identity matrix then it remains to show that

$X_2^T(I+S_2)^{-1}X_2+(0\oplus X_2-X_3)^T(I\oplus S_2+S_3)^{-1}(0\oplus X_2-X_3)$ is unchanged by permuting

ellipses 2 and 3. First compute the terms in this expression.

$$I\oplus S_2=(I+S_2^{-1})^{-1}=(S_2+I)^{-1}S_2.$$

$$0\oplus X_2=(I\oplus S_2)(0+S_2^{-1}X_2)=(S_2+I)^{-1}S_2S_2^{-1}X_2=(S_2+I)^{-1}X_2.$$

We must show $X_2^T(I+S_2)^{-1}X_2+(X_2^T(S_2+I)^{-1}-X_3^T)((S_2+I)^{-1}S_2+S_3)^{-1}((S_2+I)^{-1}X_2-X_3)$

$$=X_3^T(I+S_3)^{-1}X_3+(X_3^T(S_3+I)^{-1}-X_2^T)((S_3+I)^{-1}S_3+S_2)^{-1}((S_3+I)^{-1}X_3-X_2)$$

Simplifying this, we get an equation of the form

$$X_2^T()X_2+X_2^T()X_3+X_3^T()X_2+X_3^T()X_3 = X_2^T()X_2+X_2^T()X_3+X_3^T()X_2+X_3^T()X_3$$

It turns out that the corresponding coefficients in this quadratic are equal as is shown below.

Matching $X_2^T()X_2$ type terms requires that

$$(I+S_2)^{-1}+(S_2+I)^{-1}((S_2+I)^{-1}S_2+S_3)^{-1}(S_2+I)^{-1}=((S_3+I)^{-1}S_3+S_2)^{-1}$$

$$\Leftrightarrow \quad (S_2+I)^{-1}+(S_2+S_3(S_2+I))^{-1}(S_2+I)^{-1}=((S_3+I)^{-1}S_3+S_2)^{-1}$$

$$\Leftrightarrow \quad (S_2+S_3(S_2+I))+I=((S_3+I)^{-1}S_3+S_2)^{-1}(S_2+I)(S_2+S_3(S_2+I))$$

$$\Leftrightarrow \quad ((S_3+I)^{-1}S_3+S_2)(S_2+S_3S_2+S_3+I)=(S_2+I)(S_2+S_3S_2+S_3)$$

$$\Leftrightarrow \quad (S_3+(S_3+I)S_2)(S_2+S_3S_2+S_3+I)=(S_3+I)(S_2+I)(S_2+S_3S_2+S_3)$$

When multiplied out both sides have 12 terms which are equal. The terms of the form $X_3^T()X_3$ match by symmetry with the $X_2^T()X_2$.

To have the $X_3^T()X_2$ terms match requires that

$$\Leftrightarrow ((S_2+I)^{-1}S_2+S_3)^{-1}(S_2+I)^{-1}=(S_3+I)^{-1}((S_3+I)^{-1}S_3+S_2)^{-1}$$

$$\Leftrightarrow ((S_3+I)^{-1}S_3+S_2)(S_3+I)=(S_2+I)((S_2+I)^{-1}S_2+S_3)$$

$$\Leftrightarrow S_3+S_2(S_3+I)=S_2+(S_2+I)S_3$$

These three terms match when multiplied out. The $X_2^T()X_3$ terms match by symmetry. Since all terms match the expressions are equal.

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